J. F. Booker[6]

What follows is a rather long commentary on concepts and a very short query on possible applications.

Successful computation methods are always compromises between accuracy, generality, speed, and cost. Factors of formal elegance and intuitive appeal probably also play peripheral roles. The mobility method has proved to be a useful compromise; one hopes that the closely-related impedance method may find its niche as well.

The impedance and mobility methods form a perfect dual. Both provide for the efficient storage of bearing characteristics based on *any* suitable film model. Because pressure distributions are *not* calculated, both methods permit extremely efficient computation.

Though the methods are in one sense *inverse,* they should also be seen as *complementary* in view of their natural ranges of application: In the impedance method, instantaneous specification of eccentricity and velocity allows direct determination of force; in the mobility method, instantaneous specification of eccentricity and force allows direct determination of velocity. In appropriate applications the resulting equations of motion are in explicit form, and iterative calculations can be avoided entirely by the user in system simulation studies.

It seems productive to supplement the authors' detailed development with a concise user-oriented rationale of the two methods, proceeding from the special to the general by stages.

Special Case: Without Rotation

Impedance and mobility concepts are best understood in terms of experiments, analytical or physical, with nonrotating bearings. Such experiments attempt to relate instantaneously the journal center eccentricity (displacement) $\mathbf{e}$, squeeze velocity $\mathbf{V_S}$, and load force $\mathbf{F_L}$ or their dimensionless counterparts, eccentricity ratio ϵ, mobility $\mathbf{M}$, and impedance $\mathbf{W}$, defined by[7]

$$\epsilon = \frac{\mathbf{e}}{C}$$

$$\mathbf{M} = \frac{2\mu L}{(C/R)^3} \frac{\mathbf{V}_s}{|\mathbf{F}_L|}$$

$$\mathbf{W} = \frac{(C/R)^3}{2\mu L} \frac{\mathbf{F}_L}{|\mathbf{V}_s|}$$

so that

$$|\mathbf{W}||\mathbf{M}| = 1$$

The definitions of $\mathbf{M}$ and $\mathbf{W}$ clearly hinge on the proportionality of $|\mathbf{V_S}|$ and $|\mathbf{F_L}|$. This necessary proportionality is the *analytical* consequence of a linear Reynolds field equation with homogeneous boundary conditions and/or constraints. Alternatively, the proportionality is the *physical* consequence of an incompressible lubricant, both entering and cavitating (if at all) at near-ambient pressure.

The relation between eccentricity, velocity, and force can be displayed in "fixed" coordinates X,Y or in "moving" coordinates x,y and x',y' referenced, respectively, to velocity and force directions as shown in Fig. 14.

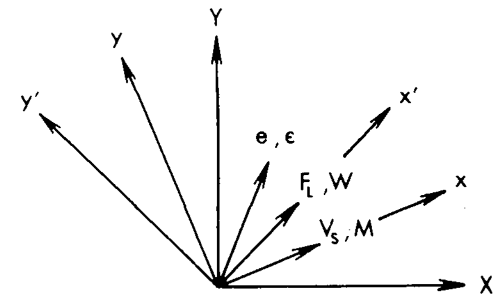

Fig. 14 Coordinate systems associated with dimensional/dimensionless eccentricity, velocity, and force vectors

Using the two "moving" frames, the same data can be displayed in alternate maps of impedance *or* mobility plotted over the clearance space of all possible eccentricity ratios. Figs. 15 and 16, kindly supplied by the authors, allow a comparison of typical impedance and mobility maps for the same basic data.[9] The two maps are oriented to velocity and force directions respectively as shown. Separate curvilinear families[11] indicate magnitude and direction of impedance and mobility vectors. Though specific to a particular length/diameter ratio,[10] such maps apply to all orientations of bearings with circumferential symmetry.

Either map, impedance or mobility, represents the complete relationship of eccentricity, velocity, and force for a particular bearing. Since the same basic data are displayed in both maps, each point on one map corresponds to a (different) point on the other,[12] and possession of one (entire) map permits construction of the other. This inverse relationship is immediately apparent along the midlines of the maps; elsewhere the connection is less obvious. The interested reader can verify that corresponding sample points are indicated in Figs. 15 and 16. As a further exercise he can examine the corresponding forms of the "equilibrium locus" whereon eccentricity and mobility (or velocity) vectors are perpendicular.

Availability of the *appropriate* map data is thus tantamount to solution of *all* problems involving the relation of eccentricity, velocity, and force for a particular non-rotating bearing. That is, specification of $\mathbf{e}$ and $\mathbf{V_s}$ allows direct determination of $\mathbf{F_L}$ via $\mathbf{W}$; alternatively, specification of $\mathbf{e}$ and $\mathbf{F_L}$ allows direct determination of $\mathbf{V_s}$ via $\mathbf{M}$. The *numerical* implementation of these procedures[13] can be described as follows:

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[7] In the notation adopted by the authors, $\mathbf{W}$ is quite *literally* the inverse of $\mathbf{M}$. However, it should be noted that $\mathbf{W}$ is *not* the inverse mobility vector $\mathbf{M}^{-1}$ introduced in the writer's discussion of the closely-related impulse method of Blok [28].[8]

[8] See Additional References below.

[9] Actually, *two* sets of basic data are illustrated in each of Figs. 15 and 16. The left half of each map is based on the Ocvirk short bearing 2π (complete) film model; the right is based on the corresponding π (cavitated) film model. Thus each map may be expected to show increasing left-right symmetry in approaching the bottom center point.

[10] For the Ocvirk short bearing film model illustrated in Figs. 15 and 16 the dependence on length/diameter ratio is strong but simple: magnitudes of impedance (or mobility) vary directly (or inversely) with its square.

[11] The direction and magnitude families are generally *not* orthogonal, except along the midlines of both maps and portions of the impedance map boundary.

[12] Exceptionally, a point of vanishing mobility corresponds to a semicircle of infinite impedance.

[13] Essentially the same numerical procedures can be used to transform *non*dimensional quantities as well. i.e., specification of ϵ and $\mathbf{M}$ allows determination of $\mathbf{W}$; alternatively, specification of ϵ and $\mathbf{W}$ allows determination of $\mathbf{M}$.

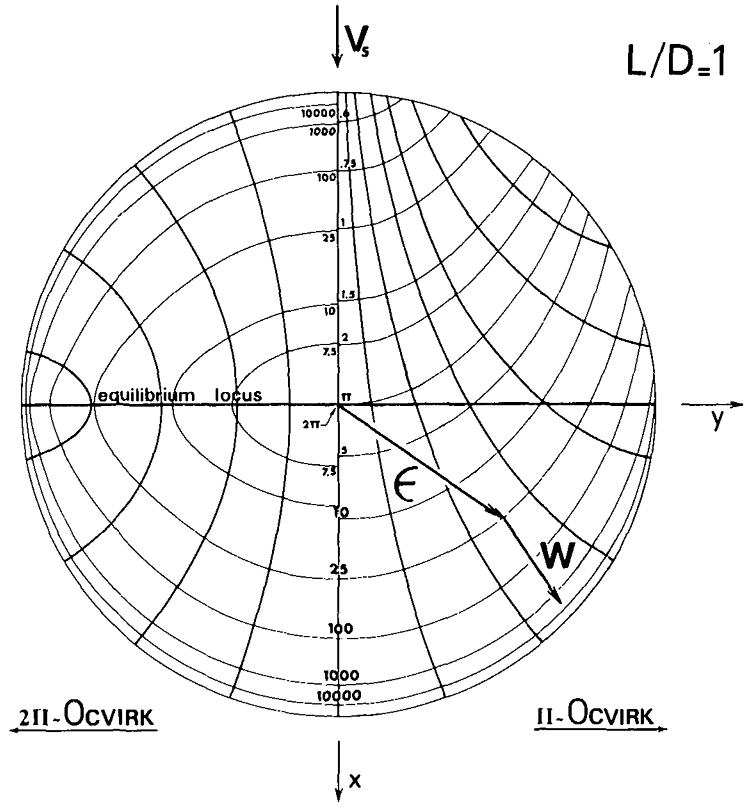

Fig. 15 Clearance circle map of impedance; short bearing solution (after Childs, Moes, and van Leeuwen)

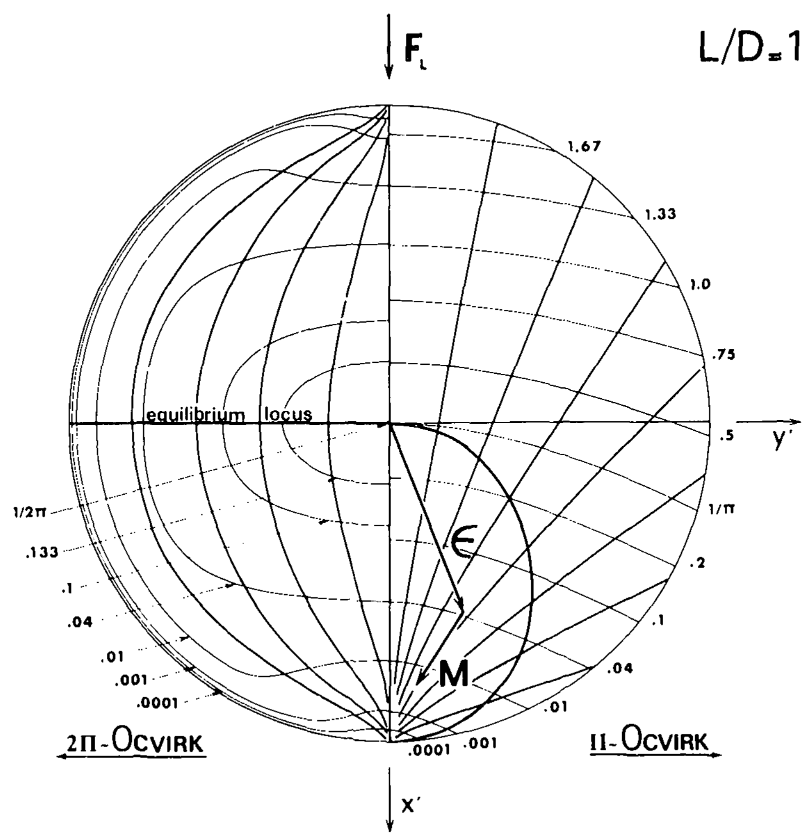

Fig. 16 Clearance circle map of mobility; short bearing solution (after Booker)

A. Impedance Method: $\mathbf{e}, \mathbf{V_S} \rightarrow \mathbf{F_L}$
Find velocity magnitude

$$|V_S| = [(V_S{}^X)^2 + (V_S{}^Y)^2]^{1/2}$$

and direction cosines

$$\begin{Bmatrix} c \\ s \end{Bmatrix} = \frac{1}{|V_S|} \begin{Bmatrix} V_S{}^X \\ V_S{}^Y \end{Bmatrix}$$

Transform eccentricity to velocity frame

$$\begin{Bmatrix} e^x \\ e^y \end{Bmatrix} = \begin{bmatrix} +c & +s \\ -s & +c \end{bmatrix} \begin{Bmatrix} e^X \\ e^Y \end{Bmatrix}$$

Find dimensionless force (impedance) in velocity (map) frame[14]

$$\begin{Bmatrix} W^x(\epsilon^x, \epsilon^y, L/D) \\ W^y(\epsilon^x, \epsilon^y, L/D) \end{Bmatrix}$$

Transform impedance to original frame

$$\begin{Bmatrix} W^X \\ W^Y \end{Bmatrix} = \begin{bmatrix} +c & -s \\ +s & +c \end{bmatrix} \begin{Bmatrix} W^x \\ W^y \end{Bmatrix}$$

Find dimensional force

$$\begin{Bmatrix} F_L{}^X \\ F_L{}^Y \end{Bmatrix} = \frac{2\mu L}{(C/R)^3} |V_S| \begin{Bmatrix} W^X \\ W^Y \end{Bmatrix}$$

B. Mobility Method: $\mathbf{e}, \mathbf{F_L} \rightarrow \mathbf{V_S}$
Find force magnitude

$$|F_L| = [(F_L{}^X)^2 + (F_L{}^Y)^2]^{1/2}$$

and direction cosines

$$\begin{Bmatrix} c' \\ s' \end{Bmatrix} = \frac{1}{|F_L|} \begin{Bmatrix} F_L{}^X \\ F_L{}^Y \end{Bmatrix}$$

Transform eccentricity to force frame

$$\begin{Bmatrix} e^{x'} \\ e^{y'} \end{Bmatrix} = \begin{bmatrix} +c' & +s' \\ -s' & +c' \end{bmatrix} \begin{Bmatrix} e^X \\ e^Y \end{Bmatrix}$$

Find dimensionless velocity (mobility) in force (map) frame[15]

$$\begin{Bmatrix} M^{x'}(\epsilon^{x'}, \epsilon^{y'}, L/D) \\ M^{y'}(\epsilon^{x'}, \epsilon^{y'}, L/D) \end{Bmatrix}$$

Transform mobility to original frame

$$\begin{Bmatrix} M^X \\ M^Y \end{Bmatrix} = \begin{bmatrix} +c' & -s' \\ +s' & +c' \end{bmatrix} \begin{Bmatrix} M^{x'} \\ M^{y'} \end{Bmatrix}$$

Find dimensional velocity

$$\begin{Bmatrix} V_S{}^X \\ V_L{}^Y \end{Bmatrix} = \frac{(C/R)^3}{2\mu L} |F_L| \begin{Bmatrix} M^X \\ M^Y \end{Bmatrix}$$

General Case: With Rotation

Extension of these impedance/mobility procedures to practical problems involving rotation of journal and/or sleeve is trivially (and surprisingly) simple:

Consider an observer fixed to the sleeve center but rotating at the average angular velocity $\bar{\omega}$ of journal and sleeve.[16,17] The actual journal center velocity $\mathbf{V}$ and the velocity $\mathbf{V_s}$ apparent to the observer are related to the journal center eccentricity $\mathbf{e}$ and the observer's angular velocity $\bar{\omega}$ by the simple kinematic expression

$$\mathbf{V} - \mathbf{V_s} = \bar{\omega} \times \mathbf{e}$$

Since the average angular velocity of journal and sleeve apparent to the observer would vanish identically, the generation of pressure and resultant force $\mathbf{F_L}$ would seem to be related solely to the apparent (squeeze) velocity $\mathbf{V_S}$ *in exactly the same way as for the non-rotating bearings considered previously.*

Thus extension of the previous numerical procedures to general problems requires only the use of the kinematic relation above in the form

$$\begin{Bmatrix} V_S{}^X \\ V_S{}^Y \end{Bmatrix} = \begin{Bmatrix} V^X \\ V^Y \end{Bmatrix} - \begin{bmatrix} 0 & -\bar{\omega} \\ +\bar{\omega} & 0 \end{bmatrix} \begin{Bmatrix} e^X \\ e^Y \end{Bmatrix}$$

before the impedance procedure, and in the form

$$\begin{Bmatrix} V^X \\ V^Y \end{Bmatrix} = \begin{Bmatrix} V_S{}^X \\ V_S{}^Y \end{Bmatrix} + \begin{bmatrix} 0 & -\bar{\omega} \\ +\bar{\omega} & 0 \end{bmatrix} \begin{Bmatrix} e^X \\ e^Y \end{Bmatrix}$$

after the mobility procedure.

A few general observations can now be made.

It is interesting to note that the *simplest* physical and analytical experiments produce mobility and impedance data respectively. Thus mobility maps (such as Fig. 16) are considerably more difficult to obtain through analytical means than are impedance maps (such as Fig. 15), and are therefore often determined from the latter as a secondary step. The preparation of either type of map is a task of considerable consequence, not likely to be undertaken for a single potential application.

In the case of the mobility map, the direction lines are in fact the path lines followed by a nonrotating journal moving in a nonrotating sleeve under the action of a nonrotating load. No analogous physical interpretation seems to arise for the impedance map. The mobility map can be used directly as the basis for a graphical determination of journal trajectories in appropriate problems. No analogous application seems obvious for the impedance map, which seems useful chiefly as a graphical summary of the numerical data it represents.

The impedance and mobility methods share limitations as well as possibilities. As noted previously, their very definitions foreclose the study of compressible films or the effect of variation of inlet pressure. Similarly, practical constraints on map development appear to limit most applications to cases with circumferential symmetry.[18] Thus little can be done with either method to study alternate inlet arrangements.

Since the mobility formulation is appropriate to cases in which instantaneous force is *known,* it has found its widest application to problems in reciprocating machinery (despite the severe limitations just cited).

Since the impedance formulation is appropriate to cases in which instantaneous force is *desired,* it seems most suited to problems in rotating machinery, particularly those involving damper bearings (thus largely evading the limitations noted). Unfortunately, however, design changes for enhanced stability of rotating machinery generally compromise bearing circumferential symmetry purposely and severely (thus imposing the limitations unavoidably).

Can the authors suggest other potential applications for the impedance (and/or mobility) method(s) which minimize the impact of *inherent limitations?*

Additional References

28 Blok, H., "Full Journal Bearings Under Dynamic Duty: Impulse Method of Solution and Flapping Action," JOURNAL OF LUBRICATION TECHNOLOGY, TRANS. ASME, Series F, Vol. 97, No. 2, Apr 1975, pp. 168–179.

29 Rohde, S. M., "Computational Techniques in the Analysis and Design of Fluid Film Bearings," General Motors Research Laboratories, Warren, Mich., Research Publication GMR-2279, Oct. 1976.

[14] Published map data is available for a variety of film models.

[15] Published map data is available for a variety of film models.

[16] The neglect of inertial effects in the derivation of the Reynolds equation assures the validity of such a rotating observer's predictions of film pressure and averaged flow.

[17] The concept of alternate observers is elaborated in the writer's paper [3], as well as in published discussion of the review by Campbell, et al [15], and in the forthcoming review by Rohde [29].

[18] Reference [26] provides a novel exception to the rule elaborated in the writer's discussion of reference [28].

P. E. Allaire[19]

The authors of this paper should be congratulated for an excellent paper transforming the mobility solution of Booker into a form more appropriate for rotor dynamics. Not only has the impedance description been developed but also it has been used for one example or another in most of the possible application areas: bearing forces, transient analysis, squeeze film dampers, dynamic coefficients, and stability. The advantages of the approach are well discussed and documented.

It appears that the major application of this type of analysis is in transient rotor dynamics. Computer time for the method is quite low, making it ideal for many repeated calculations. Thus some of the limitations encountered by Myrick and Rylander [12] in the number of nodal points at which the pressure could be evaluated can be avoided. A large portion of the paper is devoted to linearized bearing characteristics and stability but the number of times that these quantities must be evaluated for a given application is not usually large. The advantages of the more general finite differences or finite elements are likely to justify the longer running time.

One of the disavantages of the approach described here for plain journal bearings is that the combination of short and long bearing solutions has been shown to be accurate for a number of cases but there is no method presented for estimating the error. If a set of journal positions and velocities were encountered where substantial errors occured, only comparison with a different numerical or analytical finite length solution would show the error. No method analogous to adding more nodes or another term of an infinite series and observing the change in force (for example) appears to be available.

Suggested extensions of the mobility-impedance analysis procedure include lobed bearings for which all lobes are at high eccentricity ratios. Perhaps it should be pointed out that no short bearing analysis exists for partial arc or lobed bearings due to the boundary conditions at the end of the lobes. Thus the present method of a weighted combination of short and long bearing solutions is not possible.

An alternative approach to the methods discussed in this work is that of Hays [30, 31][20] where a variational principle equivalent to Reynolds' equation is minimized with an infinite trigonometric series for the pressure. This method can be generalized to the full dynamic conditions for either a squeeze film damper or plain journal bearing. Because of the relatively simple trigonometric nature of the integrals for the force on the journal, an algebraic series solution to the finite length journal bearing results. Running time is comparable to that for the short bearing. The method is also applicable to partial arc, multilobe and other types.

Additional References

30 Hays, D. F., "Squeeze Films: A Finite Journal Bearing with A Fluctuating Load," *Journal of Basic Engineering,* TRANS. ASME, Series D, Vol. 83, Dec., 1961, pp. 579–588.

31 Hays, D. F., "A Variational Approach to Lubrication Problems and the Solution of the Finite Journal Bearing," *Journal of Basic Engineering,* TRANS ASME, Series D, Vol. 81, Mar., 1959, pp. 13–23.

Author's Closure

Professor Booker's additional clarifying comments on the relationships of the mobility and impedance vectors and the basic limitations of the methods are appreciated. In response to Professor Booker's question concerning the inherent limitations of the methods, the authors suggest the following additional applications for the methods:

(*a*) **Squeeze-Film Damper Coefficients.** In addition to the damper coefficient formulas of Tonneson [9] given in equation (55), for circular orbits the stiffness and damping coefficients of equations (47) and (50) can be used to define "equivalent" linear stiffness and damper coefficients for a damper. Specifically, appropriate stiffness and damping coefficients are

$$K_d = a_{\epsilon\epsilon} = 2\mu L \left(\frac{R}{C}\right)^3 \epsilon_0\dot{\phi}_0 \left\{\frac{W_\epsilon(\epsilon_0, \alpha_0)}{\epsilon_0} + \frac{\partial W_\epsilon(\epsilon_0, \alpha_0)}{\partial\epsilon}\right\}$$

$$C_d = b_{\beta\beta} = 2\mu L \left(\frac{R}{C}\right)^3 |W_\beta(\epsilon_0, \alpha_0)|$$

where $\alpha_0 = \pi/2$ for clockwise rotation, and $\alpha_0 = -\pi/2$ for counterclockwise rotation. This formula for K_d differs from that employed by Cunningham, et al. [32], who use the following "average" definition, $K_d \simeq -F_\epsilon/C\epsilon_0 = 2\mu L(R/C)^3\, \dot{\phi}_0 W_\epsilon(\epsilon_0, \alpha_0)$

(*b*) **Pivoted Pad Bearings.** Impedance descriptions have been developed for the rectilinear Michell pad bearing, and will be included in a forthcoming publication. At present, impedance descriptions can be calculated for radial pivoted pads [26], which means that transient simulations may be carried out for rotors supported in pivoted-pad bearings. However, the prospects for finding accurate analytic approximations for a radial pivoted pad impedance are not encouraging, which implies that the derivation of analytic stiffness and damping coefficients for this type of bearing from impedances is also unlikely.

Professor Allaire's inquiries concerning the potential accuracy of the impedances provided for finite-length bearings can best be answered by a careful review of the methods employed for their derivation. The impedances provided are based on previously calculated mobilities, which were obtained by numerical solution of the Reynolds equation for finite-length bearings over a range of L/D ratios. The observation was made that a very accurate analytic approximation to these mobilities could be obtained from a weighted sum of short and long bearing solutions. At the equilibrium locus, the analytical impedance solutions are less than .1 percent in error when compared to impedances calculated from either finite difference or finite-element methods. The authors regret any implication that these same functions could be employed to approximate numerically calculated impedances for other types of bearings.

With regard to the method of Hays [30], [31], this method has been applied to finite length bearings for 2π and π regions of positive pressure. The application of this method for the more accurate ($p = \partial p/\partial\theta = 0$) cavitating boundary conditions appears to the authors to be a very difficult undertaking. A solution based on these boundary conditions would be required to obtain a bearing model comparable in accuracy to the finite-length cavitating impedance of equations (29)–(32).

The authors fail to see the advantages suggested by Professor Allaire for calculating stiffness and damping coefficients for a plain journal bearing by numerical methods, when compared to simply evaluating the formulas of equations (53) and Appendix B. In our opinion, the formulas can be evaluated much more quickly, and provide comparable accuracy.

Additional References

[32] Cunningham, R. E., Fleming, D. P., and Gunter E. J., "Design of a Squeeze-Flim Damper for a Multimass Flexible Rotor," *Journal of Engineering for Industry,* TRANS. ASME, Nov. 1975, pp. 1383–1389.

[19] Assistant Professor Mechanical Engineering Department, University of Virginia, Charlottesville, Va.

[20] Numbers 30–31 in brackets designate Additional References at end of discussion