As the authors have indicated, their results using the new algorithm seem to match very well with those obtained by us [Paranjpe and Goenka (1989)] for the engine main bearing application using the Elrod Algorithm and a finite difference solution. We would welcome a direct comparison of the key figures of merit which in our case were as follows:

Minimum film thickness	=	1.88 μm
Maximum film pressure	=	103.3 MPa
Average power loss	=	491.0 W
Average flow	=	15.66 cm^3/s

Have the authors made other comparisons with numerical results of the Elrod algorithm or with experimental results?

Obviously, decoupling would be needed if $\dot{\Pi}$ were to be solved for directly; otherwise additional matrix inversion would be needed at each time step. Could the authors give some details on how this decoupling might be accomplished? Could the authors also explain what a "decoupled modified Euler step" is?

In our results for the engine main bearing (both the earlier published results and the FEM implementation of the new algorithm) we have found sharp spikes in the $\dot{e}$ curves. Have the authors encountered such spikes and, if so, could they comment on them?

The ability to handle arbitrary geometry and grids is often thought of as the biggest advantage of FEM over FDM. Can this algorithm be extended to handle arbitrary grids, especially with respect to the upwinding scheme?

Once again, this work should prove very beneficial, at least to those who use FEM to solve journal bearing problems. Not only will this algorithm provide physically more realistic solutions, but we won't be surprised if it does so at a significantly lower computation cost. We certainly intend to use this algorithm for engine bearing analysis at General Motors.

Authors' Closure

Drs. Goenka and Paranjpe provide both good news and bad news. The good news, of course, is that our algorithm could be implemented by others (albeit experts) on the basis of our published description alone. The bad new is that it didn't work as perfectly as one would like.

We suspect that much of the difficulty can be traced to the time integration scheme they used, which is somewhat simpler than ours, which we have described very incompletely in Appendix B.

Generally, an integration scheme must be both stable and accurate for reasonable time steps. The simplest choice, Euler's method, is neither. For direct problems (specified kinematics), the *modified* Euler method (a 2nd order Runge-Kutta method related to the trapezoidal rule and requiring 2 derivative evaluations per time step) works quite well for the prediction of nodal densities. (See, for example, Conte and de Boor (1980).)

For indirect problems (specified kinetics), however, we have the additional complication that the rigid body displacement must be predicted more-or-less *simultaneously* with the densities. For simplicity, however, we "decouple" the two integration procedures by performing them *sequentially*; short time steps provide justification for the approximation. See Kumar (1991) for further details of the procedure.

More awkward is the request for quantitative comparison with the results of Paranjpe and Goenka (1990), which we would expect to be very accurate. Our own results follow:

Minimum film thickness	=	1.325 μm
Maximum film pressure	=	114.1 MPa
Average power loss	=	(not computed)
Average flow	=	15.67 cm^3/s

The significant discrepancies in cycle extrema are unexplained; it is unfortunate that we computed only one cycle average (for which agreement is little short of miraculous).

In a later paper (Kumar and Booker (1991)) we have made detailed comparisons with both numerical results of the Elrod algorithm and with experimental results. The latter study reports and explains in detail very sudden motions as described by Drs. Goenka and Paranjpe, showing them *not* to be numerical artifacts.

We see no inherent reason why the present algorithm could not be applied to arbitrary grids, though we have not done so. In particular, the "upwinding" scheme poses no obvious problem.

In the "upwind" computation strategy for the (flow) vector reflecting "Couette" flow, the upwind node is that most upwind (upstream) with respect to the relevant surface velocity component. In the case of ties, the node closer to the centroid would be chosen. (In the regular grids of right triangles which we have studied exclusively, this has meant the node at the right angle.)

There are *theoretical* grounds for applying a similar "upwind" density sampling in evaluation of the (fluidity) matrix reflecting "Poiseuille" flow constituents. *Practical* grounds seem lacking, however (owing to uniformity of density in relevant regions), and the algorithm as described here does not have this feature. Similarly, "upwinding" is omitted in the computation of the squeeze flow vector.

Initial and boundary conditions proposed in the Problem Formulation (and amplified in footnotes 4 and 5) continue to vex both readers and authors. It appears that density must be specified only on boundary segments where mass flux is inward (even though such boundary segments may be impossible to identify *a priori*). In our numerical algorithm, however, we simply specify density on the entire boundary and rely on the upwinding scheme to select out the necessary information.

Drs. Goenka and Paranjpe were evidently able to follow our original explanation of the algorithm; we can offer other readers some further clarification in the form of an additional table. Table C1 explains the core of the algorithm for the direct problem (specific kinematics) as laid out previously in the partitions of Table 2 and the procedure of Table 3. Note that

Table C1 Equations and Unknowns

state: ρ

equations: $\mathbf{n} = \mathbf{n_{1a}} + \mathbf{n_{1b}} + \mathbf{n_2} + \mathbf{n_3}$

$$\{q\} = [K]\{p\} + \{Q^U\} + \{Q^{\dot{h}}\} + \{Q^{\dot{\rho}}\}$$

$$\begin{Bmatrix} \mathbf{q_{1a}} \\ \hline q_{1b} \\ \hline q_2 \\ \hline \mathbf{q_3} \end{Bmatrix} = \left[\begin{array}{c|c|c|c} K_{1a1a} & K_{1a1b} & K_{1a2} & K_{1a3} \\ \hline K_{1b1a} & K_{1b1b} & K_{1b2} & K_{1b3} \\ \hline K_{21a} & K_{21b} & K_{22} & K_{23} \\ \hline K_{31a} & K_{31b} & K_{32} & K_{33} \end{array}\right] \begin{Bmatrix} \mathbf{p_{1a}} \\ \hline p_{1b} \\ \hline p_2 \\ \hline p_3 \end{Bmatrix}$$

$$+ \begin{Bmatrix} Q^U{}_{1a} \\ \hline Q^U{}_{1b} \\ \hline Q^U{}_2 \\ \hline Q^U{}_3 \end{Bmatrix} + \begin{Bmatrix} Q^{\dot{h}}{}_{1a} \\ \hline Q^{\dot{h}}{}_{1b} \\ \hline Q^{\dot{h}}{}_2 \\ \hline Q^{\dot{h}}{}_3 \end{Bmatrix} + \begin{Bmatrix} Q^{\dot{\rho}}{}_{1a} \\ \hline \mathbf{Q^{\dot{\rho}}{}_{1b}} \\ \hline \mathbf{Q^{\dot{\rho}}{}_2} \\ \hline Q^{\dot{\rho}}{}_3 \end{Bmatrix}$$

unknowns: $\mathbf{n} = \mathbf{n_{1a}} + \mathbf{n_{1b}} + \mathbf{n_2} + \mathbf{n_3}$

$\{\mathbf{p_{1a}}\}$

$\{\mathbf{Q^{\dot{\rho}}{}_{1b}}\} \rightarrow \{\dot{\rho}_{1b}\}$

$\{\mathbf{Q^{\dot{\rho}}{}_2}\} \rightarrow \{\dot{\rho}_2\}$

$\{\mathbf{q_3}\}$

constraints: $\mathbf{n_1} = \mathbf{n_{1a}} + \mathbf{n_{1b}}$

$\{p_{1a}\} \geq \{p_{cav}\}$

$\{\dot{\rho}_{1b}\} < \{0\}$

determination of p_{1a} requires matrix decomposition (of K_{1a1a}); subsequent determination of other unknowns requires matrix multiplication only.

Additional References include comprehensive reviews by Dowson and Taylor (1979) and by Brewe, Ball, and Khonsari (1990), as well as a finite difference implementation of the Elrod algorithm by Dowson, Miranda, and Taylor (1984). We also list three closely related (and unavoidably overlapping) reports of various applications of the present algorithm by Kumar and Booker (1990), Booker and Kumar (1991), and Kumar and Booker (1991), all of which are based on the recent thesis by Kumar (1991).

Interested readers should particularly note other related work reported contemporaneously with our own. In finite difference implementations of Elrod algorithm variations, Vijayaraghavan and Keith (1990a,b,c) continue their development and application of modern numerical methods, while Han and Paranjpe (1990) extend their work to thermohydrodynamic analysis. Bayada, Chambat, and El Alaoui (1990) provide new variational formulations implemented by finite element algorithms.

Additional References

Dowson, D., and Taylor, C. M., 1979, "Cavitation in Bearings," *Annual Review of Fluid Mechanics*, Annual Reviews, Palo Alto, CA, Vol. 11, pp. 35–66.

Conte, S.D., and de Boor, C., 1980, *Elementary Numerical Analysis*, 3rd Edition, McGraw-Hill.

Dowson, D., Miranda, A. A. S., and Taylor, C. M., 1984, "Implementation of an Algorithm Enabling the Determination of Film Rupture and Reformation Boundaries in a Liquid Film Bearing," *Developments in Numerical and Experimental Methods Applied to Tribology, Proceedings 10th Leeds-Lyon Symposium on Tribology*, D. Dowson, C. M. Taylor, M. M. Godet, and D. Berthe, eds., Butterworths, London, England, pp. 60–70.

Brewe, D. E., Ball, J. H., and Khonsari, M. M., eds., 1990, *Current Research in Cavitating Fluid Films*, Special Publication SP-28, STLE.

Vijayaraghavan, D., and Keith, T. G., Jr., 1990, "An Efficient, Robust, and Time Accurate Numerical Scheme Applied to a Cavitation Algorithm," ASME JOURNAL OF TRIBOLOGY, Vol. 112, No. 1, Jan., pp. 44–51

Vijayaraghavan, D., and Keith, T. G., Jr., 1990, "Grid Transformation and Adaption Techniques Applied in the Analysis of Cavitated Journal Bearings," ASME JOURNAL OF TRIBOLOGY, Vol. 112, No. 1, Jan., pp. 52–59.

Vijayaraghavan, D., and Keith, T. G., Jr., 1990, "Analysis of a Finite Grooved Misaligned Journal Bearing Considering Cavitation and Starvation Effects," ASME JOURNAL OF TRIBOLOGY, Vol. 112, No. 1, Jan., pp. 60–67.

Bayada, G., Chambat, M., and El Alaoui, M., 1990, "Variational Formulations and Finite Element Algorithms for Cavitation Problems," ASME JOURNAL OF TRIBOLOGY, Vol. 112, No. 2, Apr., pp. 398–403.

Han, T., and Paranjpe, R. S., 1990, "A Finite Volume Analysis of the Thermohydrodynamic Performance of Finite Journal Bearings," ASME JOURNAL OF TRIBOLOGY, Vol. 112, No. 3, July, pp. 557–566.

Kumar, A., and Booker, J. F., 1990, "Cavitation and Thermal Effects in Journal Bearing Applications," *Proc. Japan International Tribology Conference*, Japan Society of Tribologists, Tokyo, Japan, Vol. III, pp. 1491–1496.

Booker, J. F., and Kumar, A., 1991, "Mass-Conservative Cavitation Analysis for Engine Bearings," *Vehicle Tribology, Proceedings 17th Leeds-Lyon Symposium on Tribology*, Leeds, England, Sept. 4–7, 1990, D. Dowson, C. M. Taylor, and M. M. Godet, eds., Elsevier, Amsterdam, The Netherlands.

Kumar, A., 1991, *Mass Conservation in Cavitating Bearings: A Finite Element Approach*, Ph.D. thesis, Cornell University, Ithaca, NY, Jan.

Kumar, A., and Booker, J. F., 1991, "A Finite Element Cavitation Algorithm: Application/Validation," ASME JOURNAL OF TRIBOLOGY, Vol. 113, No. 2, Apr., pp. 255–261.