APPENDIX A

Finite Element Details

To simplify notation the summation sign has been dropped and, unless otherwise specified, summation is from 1 to n_e over repeated (dummy) subscripts.

Within each element,[6] we choose to relate the distribution of the variables to their n_e nodal values by n_e shape functions N_i so that

$$\begin{aligned} p &= N_i p_i \\ \mathbf{U} &= N_i \mathbf{U}_i \\ \partial h/\partial t &= N_i \dot{h}_i \end{aligned} \tag{5}$$

With this assumption, the fluidity matrix is

$$\begin{aligned} K_{ij} &\equiv -\int_A \frac{\rho h^3}{12\mu} \nabla N_i \cdot \nabla N_j dA \\ &= -\int_A \frac{\rho h^3}{12\mu} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dA \end{aligned} \tag{6}$$

and the *flows* are defined below.

Special approximations seem necessary to account satisfactorily for average density distribution. Shear flow Q^U utilizes *upwinding* to represent downwind mass transport due to tangential surface motion.

Shear flow is

$$\begin{aligned} Q_i^U &\equiv \int_A \rho h \nabla N_i \cdot \mathbf{U} dA \\ &= Q_i^{Ux} + Q_i^{Uy} \end{aligned} \tag{7}$$

where

$$\begin{aligned} Q_i^{Ux} &= \int_A \rho h \frac{\partial N_i}{\partial x} U^x dA \\ &\approx \rho_k \int_A h \frac{\partial N_i}{\partial x} U^x dA \\ &= \rho_k U_j^x \int_A h \frac{\partial N_i}{\partial x} N_j dA \end{aligned} \tag{8}$$

is a consequence of shear action due to surface motion in the x direction where k is the node on the element farthest in the upwind direction. Flow Q_i^{Uy} is defined similarly.

Mass loss rate is

$$\begin{aligned} Q_i^{\Pi} &\equiv -\int_A \partial(\rho h)/\partial t \, N_i dA \\ &= Q_i^{\dot{\rho}} + Q_i^{\dot{h}} \end{aligned} \tag{9}$$

[6]For greater detail on the element formulation see Booker and Huebner (1972) and Kumar (1988) Appendix A.

where

$$\begin{aligned} Q_i^{\dot{\rho}} &= -\int_A h \partial\rho/\partial t N_i dA \\ &\approx -\dot{\rho}_i/n_e \int_A h dA \end{aligned} \tag{10}$$

is a consequence of density variation, and

$$\begin{aligned} Q_i^{\dot{h}} &= -\int_A \rho \partial h/\partial t N_i dA \\ &\approx -\langle\rho\rangle \dot{h}_j \int_A N_i N_j dA \end{aligned} \tag{11}$$

is a consequence of squeeze action.

Density rate flow $Q^{\dot{\rho}}$ utilizes *decoupling* to represent lack of side flow in uniform pressure (e.g., incomplete film) regions when there is no tangential motion. (Note that calculation of $\dot{\rho}$ from $Q^{\dot{\rho}}$ is trivial with this decoupled approximation.)

Total power dissipation rate (Booker (1989)) is given by

$$H \equiv H_{\text{Couette}} + H_{\text{Poiseuille}} \geq 0 \tag{12}$$

where

$$\begin{aligned} H_{\text{Couette}} &= \int_A (\mu/h) \Delta\mathbf{U} \cdot \Delta\mathbf{U} dA \geq 0 \\ H_{\text{Poiseuille}} &= \int_A (h^3/12\mu) \nabla p \cdot \nabla p dA \geq 0 \end{aligned} \tag{13}$$

These expressions are particularly useful adjuncts to finite element computations, since they afford simple evaluation on an element-by-element basis.

APPENDIX B

Time March

For direct problems, we note that the dynamic state of the system can be characterized by either film density ρ or the area mass density Π ($= \rho h$). The product Π is well behaved even in (physically-realizable) circumstances in which its components are not; this offers both conceptual and computational advantages. For example, if pressure is uniform, the area mass density Π is spatially constant in the steady state; if pressure is uniform, the product is temporally constant in the absence of tangential surface motion. (These analytical results provide useful checks and interpretations of numerical results of simple test cases.) Incrementing of ρ can be done directly by using $\dot{\rho}$; we found (at least for simple cases) that much bigger time steps could be used if the quantity Π is incremented by using $\dot{\Pi}$ (with ρ found subsequently).

For indirect problems the dynamic state of the system comprises both film density ρ (or Π) and generalized coordinate vector ϵ. For updating ϵ, we essentially assume the density ρ to be constant over a time step. At the end of each time step a decoupled modified Euler step is used to update ρ (or Π).

DISCUSSION

P. K. Goenka[1] and R. S. Paranjpe[1]

Congratulations are in order for what appears to be the first mass conserving model for cavitation which is applicable to a finite element formulation. We found the algorithm easy to implement and were able to do so in our own, finite element computer code [Goenka (1984)] in just a few hours. In our implementation we have calculated $\dot{\rho}$ and $\dot{e}$ separately and have used basic Euler stepping for time integration. We were able to duplicate the authors' results for the simpler cases of steady load and pure squeeze. However, we encountered some difficulty in duplicating the engine main bearing case. The film thickness approached zero and near 360 deg crank angle because of extremely high $\dot{e}$, and therefore, the algorithm did not converge. Did the authors encounter any such difficulty? Could they comment on probable causes for such problems and possible fine tuning of the algorithm to get around them.

[1]Engine Research Department, General Motor Research Laboratories, Warren, MI, 48090.

As the authors have indicated, their results using the new algorithm seem to match very well with those obtained by us [Paranjpe and Goenka (1989)] for the engine main bearing application using the Elrod Algorithm and a finite difference solution. We would welcome a direct comparison of the key figures of merit which in our case were as follows:

Minimum film thickness	= 1.88 μm
Maximum film pressure	= 103.3 MPa
Average power loss	= 491.0 W
Average flow	= 15.66 cm^3/s

Have the authors made other comparisons with numerical results of the Elrod algorithm or with experimental results?

Obviously, decoupling would be needed if $\dot{\Pi}$ were to be solved for directly; otherwise additional matrix inversion would be needed at each time step. Could the authors give some details on how this decoupling might be accomplished? Could the authors also explain what a "decoupled modified Euler step" is?

In our results for the engine main bearing (both the earlier published results and the FEM implementation of the new algorithm) we have found sharp spikes in the $\dot{e}$ curves. Have the authors encountered such spikes and, if so, could they comment on them?

The ability to handle arbitrary geometry and grids is often thought of as the biggest advantage of FEM over FDM. Can this algorithm be extended to handle arbitrary grids, especially with respect to the upwinding scheme?

Once again, this work should prove very beneficial, at least to those who use FEM to solve journal bearing problems. Not only will this algorithm provide physically more realistic solutions, but we won't be surprised if it does so at a significantly lower computation cost. We certainly intend to use this algorithm for engine bearing analysis at General Motors.

Authors' Closure

Drs. Goenka and Paranjpe provide both good news and bad news. The good news, of course, is that our algorithm could be implemented by others (albeit experts) on the basis of our published description alone. The bad new is that it didn't work as perfectly as one would like.

We suspect that much of the difficulty can be traced to the time integration scheme they used, which is somewhat simpler than ours, which we have described very incompletely in Appendix B.

Generally, an integration scheme must be both stable and accurate for reasonable time steps. The simplest choice, Euler's method, is neither. For direct problems (specified kinematics), the *modified* Euler method (a 2nd order Runge-Kutta method related to the trapezoidal rule and requiring 2 derivative evaluations per time step) works quite well for the prediction of nodal densities. (See, for example, Conte and de Boor (1980).)

For indirect problems (specified kinetics), however, we have the additional complication that the rigid body displacement must be predicted more-or-less *simultaneously* with the densities. For simplicity, however, we "decouple" the two integration procedures by performing them *sequentially*; short time steps provide justification for the approximation. See Kumar (1991) for further details of the procedure.

More awkward is the request for quantitative comparison with the results of Paranjpe and Goenka (1990), which we would expect to be very accurate. Our own results follow:

Minimum film thickness	= 1.325 μm
Maximum film pressure	= 114.1 MPa
Average power loss	= (not computed)
Average flow	= 15.67 cm^3/s

The significant discrepancies in cycle extrema are unexplained; it is unfortunate that we computed only one cycle average (for which agreement is little short of miraculous).

In a later paper (Kumar and Booker (1991)) we have made detailed comparisons with both numerical results of the Elrod algorithm and with experimental results. The latter study reports and explains in detail very sudden motions as described by Drs. Goenka and Paranjpe, showing them *not* to be numerical artifacts.

We see no inherent reason why the present algorithm could not be applied to arbitrary grids, though we have not done so. In particular, the "upwinding" scheme poses no obvious problem.

In the "upwind" computation strategy for the (flow) vector reflecting "Couette" flow, the upwind node is that most upwind (upstream) with respect to the relevant surface velocity component. In the case of ties, the node closer to the centroid would be chosen. (In the regular grids of right triangles which we have studied exclusively, this has meant the node at the right angle.)

There are *theoretical* grounds for applying a similar "upwind" density sampling in evaluation of the (fluidity) matrix reflecting "Poiseuille" flow constituents. *Practical* grounds seem lacking, however (owing to uniformity of density in relevant regions), and the algorithm as described here does not have this feature. Similarly, "upwinding" is omitted in the computation of the squeeze flow vector.

Initial and boundary conditions proposed in the Problem Formulation (and amplified in footnotes 4 and 5) continue to vex both readers and authors. It appears that density must be specified only on boundary segments where mass flux is inward (even though such boundary segments may be impossible to identify *a priori*). In our numerical algorithm, however, we simply specify density on the entire boundary and rely on the upwinding scheme to select out the necessary information.

Drs. Goenka and Paranjpe were evidently able to follow our original explanation of the algorithm; we can offer other readers some further clarification in the form of an additional table. Table C1 explains the core of the algorithm for the direct problem (specific kinematics) as laid out previously in the partitions of Table 2 and the procedure of Table 3. Note that

Table C1 Equations and Unknowns

state: ρ

equations: $\mathbf{n} = \mathbf{n}_{1a} + \mathbf{n}_{1b} + \mathbf{n}_2 + \mathbf{n}_3$

$$\{q\} = [K]\{p\} + \{Q^U\} + \{Q^{\dot{\Pi}}\} + \{Q^{\dot{\rho}}\}$$

$$\left\{\begin{array}{c} \mathbf{q}_{1a} \\ \hline q_{1b} \\ \hline q_2 \\ \hline \mathbf{q}_3 \end{array}\right\} = \left[\begin{array}{c|c|c|c} K_{1a1a} & K_{1a1b} & K_{1a2} & K_{1a3} \\ \hline K_{1b1a} & K_{1b1b} & K_{1b2} & K_{1b3} \\ \hline K_{21a} & K_{21b} & K_{22} & K_{23} \\ \hline K_{31a} & K_{31b} & K_{32} & K_{33} \end{array}\right] \left\{\begin{array}{c} \mathbf{p}_{1a} \\ \hline p_{1b} \\ \hline p_2 \\ \hline p_3 \end{array}\right\}$$

$$+ \left\{\begin{array}{c} \mathbf{Q}^U{}_{1a} \\ \hline Q^U{}_{1b} \\ \hline Q^U{}_2 \\ \hline Q^U{}_3 \end{array}\right\} + \left\{\begin{array}{c} \mathbf{Q}^{\dot{\Pi}}{}_{1a} \\ \hline Q^{\dot{\Pi}}{}_{1b} \\ \hline Q^{\dot{\Pi}}{}_2 \\ \hline Q^{\dot{\Pi}}{}_3 \end{array}\right\} + \left\{\begin{array}{c} \mathbf{Q}^{\dot{\rho}}{}_{1a} \\ \hline \mathbf{Q}^{\dot{\rho}}{}_{1b} \\ \hline \mathbf{Q}^{\dot{\rho}}{}_2 \\ \hline Q^{\dot{\rho}}{}_3 \end{array}\right\}$$

unknowns: $\mathbf{n} = \mathbf{n}_{1a} + \mathbf{n}_{1b} + \mathbf{n}_2 + \mathbf{n}_3$

$\{\mathbf{p}_{1a}\}$

$\{\mathbf{Q}^{\dot{\rho}}{}_{1b}\} \rightarrow \{\dot{\rho}_{1b}\}$

$\{\mathbf{Q}^{\dot{\rho}}{}_2\} \rightarrow \{\dot{\rho}_2\}$

$\{\mathbf{q}_3\}$

constraints: $\mathbf{n}_1 = \mathbf{n}_{1a} + \mathbf{n}_{1b}$

$\{p_{1a}\} \geq \{p_{cav}\}$

$\{\dot{\rho}_{1b}\} < \{0\}$