which result in mechanisms with deviations as shown in Figs. 8(a) and 8(b). The other pseudo optimal deviation is characterized by

$$\cos\mu\,|_{\theta=\theta_0+\Delta\theta_4} = -\cos\mu\,|_{\theta=0} \text{ or } -\cos\mu\,|_{\theta=\pi}.$$

These also yield a similar result. We have previously established that a mini-max deviation of the type shown in Fig. 2(d) or 2(e) (i.e., case C) is generally not possible for four precision-point function generators.

We can now summarize the totality of solutions for four precision-point function generators as follows. For a mini-max (or pseudo mini-max) transmission angle deviation in the range $\theta_0 \leq \theta \leq \theta_0 + \Delta\theta_4$, we have at most $4 + 8 + 8 = 20$ solutions corresponding to Figs. 2(b); 8(a), 8(b); and the analog of Figs. 8 (not shown). In the range $0 \leq \theta \leq 2\pi$, however, we have at most four solutions which correspond to the deviation shown in Fig. 2(a).

Conclusion

We have established that several types of mini-max deviations of the transmission angle about 90 deg are possible in the case of four-bar function generators. For these cases we have also given design procedures for synthesizing three and four precision-point function generators. These design methods can be considered to be extensions of the classical precision-point theory.

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DISCUSSION

F. Freudenstein[3]

This paper represents a thoughtful and significant contribution to the synthesis of four-bar linkages with prescribed corresponding shaft rotations and optimization of transmission-angle variation.

Among other things, the author has effectively used the theory of algebraic curves to determine the number of nontrivial solutions to the problem.

If the crank rotation is unlimited, the maximum transmission-angle deviation from a right angle may occur outside the range of the function-generation. This and the possibility of a least-square approximation technique are conceivable subjects for future research on the part of the author, whose powerful techniques could, I believe, be extended for these purposes. In the former case, a parametric representation of the link lengths (see "The Classical Transmission-Angle Problem" by F. Freudenstein and E. J. F. Primrose, *Proc. Conf. Mech.*, Institute of Mechanical Engineers, 1973, pp. 105–110) may be helpful.

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T. E. Shoup[4]

The author is to be congratulated for having presented an interesting method for designing four-bar function generators with optimum transmission angle behavior.

The author has clearly identified the various categories of mini-max situations that can occur for transmission angle. Another viewpoint on this procedure comes through an inspection of the transmission angle as a function of input angle for the various Grashof classes. This information is illustrated in Fig. 9 and indicates which categories of devices are best suited for a particular type of transmission behavior.

In the introduction to his paper, the author comments that he has been unable to find previously published analytical solutions for optimum transmission design for function-generating four-bar mechanisms. Although it was treated in less detail, a similar problem was investigated by Shoup and Pelan [10]. This effort was directed toward achieving simultaneous optimum transmission angle and optimum structural error behavior. This technique was later extended by Dukkipati and Soni [11] to include the spherical four-bar linkage. These analytical investigations were both based on the solution of nonlinear, algebraic equations and did not provide information about the number of solutions that exist in a given category. The author's elucidation of the loci of optimum transmission angle represents a unique contribution to the field of mechanism design and should be of considerable use to practitioners in this area. The discusser hopes that the author will extend his work to cover the loci associated with the transmission behavior of three-dimensional linkages.

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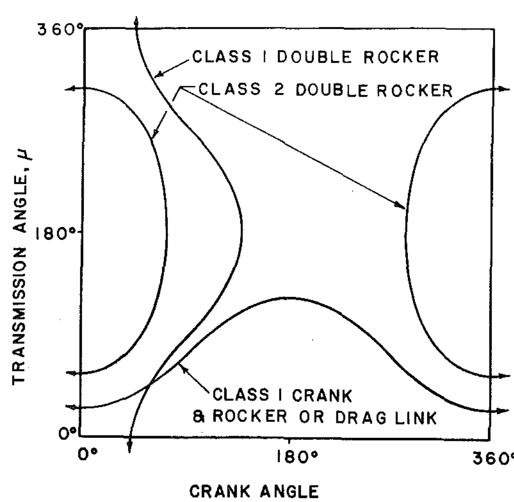

Fig. 9

Author's Closure

The author would like to thank Professors Freudenstein and Shoup for their generous comments on the paper and for their valuable suggestions for future research.

In selecting the particular problem of optimizing the transmission angle in a Chebyshev (or, mini-max) sense for a four-bar precision point function generator and the approach to study this problem, the author was primarily guided by simplicity and the generality of the resulting theory. The author, however, is aware of the possible alternate formulations and extensions, some of which are also pointed out by Professors Freudenstein (viz the case of unlimited crank rotation and a least-square formulation) and Shoup (viz a three-dimensional analog). It is hoped that meaningful results will be obtained for these cases in the future.

The author wishes to clarify his remark in the introduction that he has been unable to find previously published analytical solutions for optimum transmission design of function-generating four-bar mechanisms. The term "analytical solution" was used by the author in the context of presenting a theory for this particular optimum problem and attention was drawn to the lack of theoretical results for this problem. In particular, the author's reference was to the algebraic-geometric approach, which is typical of the classical precision position synthesis theory, and not to the numerical or nonlinear programming techniques for solving the optimum problem iteratively. These latter methods, however, have grown in popularity because they can be applied to rather complex mechanism design situations and can handle practical design constraints as well. The works referred to by Professor Shoup (references [10, 11]) and others, some of which are also listed in the review papers (references [12–14]) should be consulted for recent developments in this area. The author believes that Fig. 9 which depicts equation (6) for different mechanism classifications may provide some intuitive feel for the optimum problem.

The paper by Freundenstein and Primrose (mentioned within the text of the discussion by Freudenstein and alluded to several more times in this closure) should also be noted for its contribution, although it deals with a somewhat limited problem (references [1, 3, 5, 7, 8, 9]) compared to the one considered by the author. This paper (the one referred to within the text of Freudenstein's discussion) deals with the problem of designing optimum quick-returning crank-rocker mechanisms when the time ratio and the rocker swing angle are specified. Professor Freudenstein has suggested that their algebraic approach of finding a rational parametric representation for the squares of link lengths for this special case of four precision point function generator may be extended to the general case when the precision points are distinct and finitely separated (which was the case considered by the author).

The author is not entirely convinced of this possibility. First, if the representation of reference (Freudenstein's reference) is used, it will introduce unrelated quantities, viz time ratio and the rocker swing angle, into the general four precison point function generator problem, and this seems unnecessary. Second, if an equivalent representation is attempted directly for the general case, it is easily seen in view of equations (5), (14), and (18) that a rational parametric representation for the squares of link lengths can be found for a four precision point function generator if, and only if, the Burmester cubic given by equation (18) becomes a rational cubic (i.e., its genus is zero) or it degenerates into simpler forms. Such was precisely the case in the reference in Freudenstein's discussion where the cubic degenerated into a circle and a straight-line (references [1, 3]). In general, however, the Burmester cubic given by equation (18) is not rational (i.e., its genus is 1), and therefore a parametric representation for the squares of link lengths can be found only in terms of the doubly periodic elliptic functions (reference [15]). It seems that such a parametric representation (if it is found) will not be practical to use in transmission angle optimization.

In conclusion, the author wishes to express his appreciation to Professors Freudenstein and Shoup for writing interesting and thought-provoking discussions on the paper.

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