in which

$$H = 2\sqrt{\frac{R'}{h_0}} \arctan\left(\frac{R'}{h_0} \cdot \Phi\right) \tag{3}$$

The neutral point is found by the requirement

$$s_n^{\ +} = s_n^{\ -} \tag{4}$$

from which

$$H_n = \frac{H_i}{2} - \frac{1}{2\mu} \ln\left\{\frac{h_i}{h_0}\left(\frac{1 - \dfrac{\sigma_0}{k_0}}{1 - \dfrac{\sigma_i}{k_i}}\right)\right\} \tag{5}$$

I take the liberty to extend this consideration and to introduce a perfectly logical hypothesis: *Maximum reduction takes place if all friction force is used to pull the strip in between the rolls.* This can be formulated by

$$H_n = 0 \tag{6}$$

Using (5) and (3), we derive from (6):

$$\mu = \tfrac{1}{2}\sqrt{\frac{h_0}{R'}} \cdot \frac{\ln\left\{\dfrac{h_i}{h_0}\left(\dfrac{1 - \dfrac{\sigma_0}{k_0}}{1 - \dfrac{\sigma_i}{k_i}}\right)\right\}}{\arctan\left(\dfrac{h_i}{h_0} - 1\right)} \tag{7}$$

If no front or back tensions are applied, we find

$$\mu = \tfrac{1}{2}\sqrt{\frac{h_0}{R'}} \cdot \frac{\ln \dfrac{h_i}{h_0}}{\arctan\left(\dfrac{h_i}{h_0} - 1\right)} \tag{8}$$

This is exactly formula (7) of [1], which, however, was derived from a totally different consideration, viz., an energy approach.

For a given value of μ, we may also derive maximum and minimum possible reductions from (5) and (6) by successive approximations and find results which are similar to those mentioned in [1] and [2].

The theory of Bland and Ford is generally recognized as a valuable contribution to the mechanics of metalworking processes and is, e.g., quoted by Johnson and Mellor [5]. Therefore, it seems hardly justifiable to classify a theory which differs in approach but not in ultimate result as "dangerously misleading."

The second comment quoted seems correct to this extent that all conclusions of Avitzur could have been drawn from earlier theories. It should be added, though, that this has not been done.

However, as for reference [3], the importance is far beyond the scope of solving a strip rolling problem. It is a demonstration of a technical application of the important lower upper-bound theorem, and up to now only few scientists have been able to use this theorem successfully in this field. In my opinion, further application of extremum principles in plasticity mechanics will prove to be very useful and give a better insight in many processes which are nowadays considered to be insoluble. Therefore, as far as I am concerned, Mr. Avitzur is congratulated with his work.

Nomenclature

h = thickness of strip
k = maximum shear stress
s = normal roll pressure
R' = radius of deformed arc of contact
Φ = angular coordinate, rad
μ = coefficient of friction between roll surface and material
σ = tensile stress (front or back pull)

Suffixes

i = entry
0 = exit
n = neutral point
$+$ = between 0 and n
$-$ = between n and i

References

1 Avitzur, B., "Maximum Reduction in Cold Strip Rolling," *Proceedings of the Institute of Mechanical Engineers*, Vol. 174, No. 32, 1960, pp. 865–884.

2 Avitzur, B., "Power Analysis of Cold Strip Rolling," Journal of Engineering for Industry, Trans. ASME, Vol. 85, Series B, Feb. 1963, pp. 77–88.

3 Avitzur, B., "An Upper-Bound Approach to Cold-Strip Rolling," Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 86, No. 1, Feb. 1964, pp. 31–46.

4 Bland, D. R., and Ford, H., "The Calculation of Roll Force and Torque in Cold Strip Rolling With Tensions," *Research on the Rolling of Strip: A Symposium of Selected Papers*, 1948–1958; B.I.S.R.A., pp. 68–77.

5 Johnson, W., and Mellor, P. B., *Plasticity for Mechanical Engineers*, van Nostrand, 1962, pp. 243–252.

DISCUSSION

J. M. Alexander[3]

In reply to Mr. Mot's brief note I have the following comments: When I used the phrase "dangerously misleading" in relation to Avitzur's solution to the cold rolling problem it was because of the tendency (which still prevails) for investigators using limit analysis techniques to approach a given problem from the upper bound only. To obtain a real estimate of the limits bounding the true solution, it is really necessary to maximize lower bound solutions as well as to minimize upper bound solutions. The fact that Avitzur's solution can be shown to give equivalent results to the Bland and Ford theory simply proves that the Bland and Ford theory also provides an upper bound solution if the elastic arcs of contact are neglected and friction acts in one direction only. The justification for using Avitzur's theory would not rest on the fact that it is a minimized upper bound solution but on the fact that it gives the same or similar results to the Bland and Ford theory which has been proven over many years. This proof rests on comparison with experimental results obtained by many investigators, notably Ford himself.

It also seems rather a waste of effort to seek limit analysis solutions to problems for which realistic and satisfactory solutions already exist. The best justification for the use of limit analysis techniques lies in their application to problems of complicated deformation for which no well-defined theory exists, for example extrusion and forging. Even in such situations investigators should, in my opinion, attempt to obtain lower bound solutions, although this is rarely done. In this context I would draw Mr. Mot's attention to references [6] and [7] at the end of this discussion in which I have considered the limit analysis of extrusion and hot rolling, and tried to obtain lower bounds to give a correct "feel" for the problem.

To summarize, one could apply the phrase "dangerously misleading" to any limit analysis solution which relies only on obtaining and minimizing an upper bound. In retrospect I now wish I had used the phrase "well-meaning but unnecessary" be-

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cause I have nothing but respect for the motives behind Dr. Avitzur's work. Since that time he has written a great many papers on metal working problems and contributed a great deal of useful analytical work notably on extrusion, hydrostatic extrusion, and similar processes. He is obviously a dedicated and diligent research worker, but his analyses would gain if they were extended to obtaining maximized lower bound solutions to bracket with minimized upper bound solutions.

Additional References

6 Alexander, J. M., "On Complete Solutions for Frictionless Extrusion in Plane Strain," *Quarterly of Applied Mathematics*, Vol. 19, No. 1, p. 31.

7 Alexander, J. M., and Ford, H., "On the Limit Analysis of Hot Rolling," *Progress in Applied Mechanics*, Prager Anniversary Volume, MacMillan, 1963, p. 191.