except for a factor of $(1 - \nu^2)$. This is the same factor that applies to the beam bending deflection equation. Hence, wide beams may be treated in the usual manner, i.e., by using $E/(1 - \nu^2)$ instead of E. This is valid for all formulas given herein.

Beams of Relatively Great Depth

Classical beam theory is usually not recommended for beams having a span/depth ratio less than 8 for metal beams of compact section, or less than 15 for beams with relatively thin webs. This is "because of the importance of shear deflections." It is interesting to compare the relative magnitude of deflections due to rotation at the built-in end(s) of a beam to the deflections due to shear in the beam itself. In the case of an end loaded cantilever, both of these deflections increase linearly with the distance X from the built-in end. Hence the ratio of these deflections will be the same for all X. Fig. 4 gives the value of this ratio for rectangular beams where $E_S = E$. Shear deflections were calculated using a shear factor of 1.2 and a Poisson's ratio of 0.3. Note that for $L/h = 8$, the deflections due to rotation are an order of magnitude more significant than shear deflections, even when fillets are used.

If one takes account of the deflections due to rotation at the built-in end(s) of a beam, classical beam theory can be used for considerably shorter-stubbier beams than indicated above without introducing large errors. Classical beam theory gives satisfactory results for stresses in beams which are as short as three times the depth. Moreover, local stresses introduced by the loads in even shorter beams reduce the bending stresses below those given by simple beam theory.

References

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2 N. L. Muskhelishvili, "Some Basic Problems of the Mathematical Theory of Elasticity," P. Noordhoff Limited, Groninger, Holland, third edition, 1953, p. 471.

3 R. J. Roark, "Formulas for Stress and Strain," McGraw-Hill Book Company, Inc., New York, N. Y., third edition, 1954, p. 100.

DISCUSSION

A. H. Burr[3]

For a rectangular, cantilever beam with small fillets and a 1:1 ratio of length to depth, Fig. 3 of the paper indicates that the effect of flexibility or rotation at the built-in end is to increase the deflection at the free end by about 50 percent over that obtained from bending and shear alone. A gear tooth is a beam with about the same ratio of length to depth at the built-in end, but with a depth decreasing toward the free end. The smaller depth would increase the deflection due to bending and shear, and thus appear to give a *percentage* increase less than 50 percent due to rotation at the built-in end. Nevertheless, the increase is significant.

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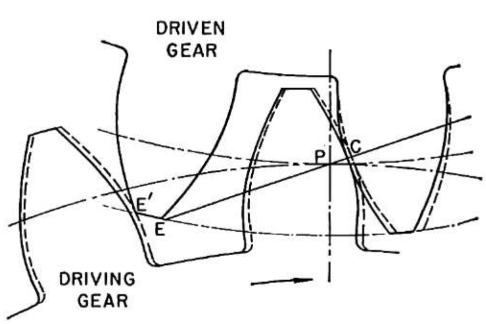

Fig. 7 The effect of tooth deflections on the engagement of gear teeth

Tooth deflection in gear drives is important because it affects the division of load between pairs of teeth and the kinematic and dynamic action. For unloaded gears, shown in full lines in Fig. 7, tooth engagement occurs at point E where the addendum circle of the driven gear intersects the theoretical line of contact EP. For loaded gears the pair of teeth already in contact at C deflects in bending. This, together with surface deformation at C, allows the driving gear to advance relative to the driven gear so that the teeth have the position shown by the dotted lines. This allows engagement to occur prematurely at point E'. As rotation continues, the teeth of the second pair are deflected and they share the load with the first pair. The distance EE' is increased by greater tooth flexibility and by errors in tooth spacing. Hence, the effect which the author of the paper has studied, the increase of deflection from rotation at the built-in end, significantly affects the tooth loading conditions.

Initial studies of load division by Peterson and Baud [4][4] and of premature engagement by Burr [5] and by Burr and Peterson [6], did not include this rotation. A recent study by Richardson [7] takes the rotation into account. He uses the tooth deflection equations developed by Weber [8] in which deformation of the rim adjacent to the root of the tooth is determined by an energy method. This method differs from the author's.

Additional References

4 R. E. Peterson and R. V. Baud, "Load and Stress Cycles in Gear Teeth," *Mechanical Engineering*, vol. 51, 1929, pp. 653–662.

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6 A. H. Burr and R. E. Peterson, "The Theoretical Aspects of Tip Relief," Semi-Annual Meeting, American Gear Manufacturers' Association, October 15, 1931.

7 H. H. Richardson, "Static and Dynamic Load, Stress, and Deflection Cycles in Spur-Gear Systems," Dynamic Analysis and Control Laboratory, Massachusetts Institute of Technology, June 30, 1958; also, "Dynamic Loading of High Speed Gearing," Trans. Seventh Conference on Mechanisms, Purdue University, October, 1962, pp. 146–160.

8 C. Weber, "The Deformation of Loaded Gears and the Effect on Their Load-Carrying Capacity, Part I," Dept. of Scientific and Industrial Research, Sponsored Research Report No. 3, London, England, 1949.

Bernard W. Shaffer[5]

In some areas of stress analysis, such as the pressure vessel field,[6] the rotational flexibility of neighboring sections is always taken into account. Yet it is frequently neglected in other fields of application because of what appears to be an obvious difference in relative rigidity. A connection between a beam and a semi-infinite body is one such application. It is, therefore, helpful that attention has been called to the fact that a significant error may be introduced by always treating such beams with the conventional built-in boundary conditions.

The author's reference to an elasticity solution based on complex variable techniques may leave some design engineers with the impression that the solution contained in the present paper satisfies all boundary conditions. Yet, as we both know, the author has not been able to satisfy continuity of displacements in three dimensions at the junction between the beam and the half-plane within the framework of the theory of elasticity. Consequently, any formula for rotation of the support can at best only be a reasonable approximation for use by the design engineer. This fact does not detract from the author's contribution because the total elasticity problem is a formidable one. Nevertheless, I hope the author agrees that the foregoing fact should be called to the attention of the reader.

[4] Numbers in brackets designate Additional References at end of discussion.

[5] Professor of Mechanical Engineering, New York University, New York, N. Y.

[6] Bernard W. Shaffer, Ira Cochin, and Morton Mantus, "Optimum Length of a Rocket Wall Extension Beyond Its Circumferential Keyway," *Transactions of the New York Academy of Sciences*, series II, vol. 21, no. 4, February, 1959, pp. 295–315.

Author's Closure

Concerning Dr. Shaffer's discussion, it should be emphasized that the usual assumption that plane sections remain plane neglects the local flexibility at the junction of neighboring sections ipso facto. This local flexibility should be added to the gross flexibility of the members themselves, whether they are beams, plates, or shells. To the author's knowledge, this has seldom been done in the analysis of pressure vessels or any other structures.

Dr. Shaffer is quite correct in his statement that the author has not proved continuity of displacements in three directions at the junction between the beam and the half-plane for the solution given by equation (1) of the paper. However, the assumption of linearly distributed stresses at the junction gives an upper bound (using the theorem of minimum energy of stresses). On the other hand, assuming linear displacements at the junction gives a lower bound (since by preventing distortion the construction is made more rigid).

Hence, as pointed out in the paper, the solution is bounded between $\frac{18\,\bar{M}}{\pi E_s(h')^2}$ and $\frac{15.18\,\bar{M}}{\pi E_s(h')^2}$. The $\frac{16.67\,\bar{M}}{\pi E_s(h')^2}$ solution given by equation (1) is then established to be accurate within 10 percent. The corresponding maximum possible error in the calculated beam stresses and deflections is always less than 10 percent of the *difference* between the solution which includes the local flexibility and that which assumes a perfectly rigid support. In the case of statically indeterminate beams, the maximum possible error is even smaller.

Professor Burr's discussion of the significance of the author's work in gear design is appreciated. It may be noted that the local flexibility at the built-in end of a gear tooth permits a rotation due to shear as well as that due to bending considered in the present paper. Further, there is a significant displacement at the built-in end of the loaded gear tooth with respect to the adjacent tooth caused by the moment and shear loads. Calculation of the free end deflections of actual gear teeth forms, including all of these effects, indicates that the increase due to the local flexibility at the built-in end is usually about 100 percent over that due to bending and shear in the "beam."

Weber obtained the solution for local deformations due to a linear bending stress distribution using complex variable techniques. This approach differs from the author's. His effective rotation, like the author's, is based on the equivalent energy concept, and is identical to the $\frac{18\,\bar{M}}{\pi E_s(h')^2}$ solution obtained by the author for a linear stress distribution.