$$\left(\frac{b}{a}\right)^2 = \frac{e^{2\bar{\epsilon}_{am}\sqrt{\frac{G}{\alpha_z}}} - 1}{e^{2\bar{\epsilon}_{bm}\sqrt{\frac{G}{\alpha_z}}} - 1}$$

The solution of equation (j') will yield a value of b/a for the specific value of n and the assumed value of $\bar{\epsilon}_{bm}$. When plotted, these values yield one point in Fig. 2. This procedure is repeated until a sufficient number of points are obtained to make possible the construction of the curve as shown in Fig. 2. Using Fig. 2, it is now possible to read off the value of $\bar{\epsilon}_{bm}$ for a specific value of b/a and the specific value of n as determined from the log-log plot.

4 Using the value of $\bar{\epsilon}_{bm}$ corresponding to a specific value of b/a, as found from Fig. 2, $\bar{\epsilon}_{am}$ can be read directly from Fig. 3. Fig. 3 is a graphical representation of equation (j'). Fig. 3 shows the relation between $\bar{\epsilon}_{bm}$ and $\bar{\epsilon}_{am}$ for various values of b/a as the parameter.

With these known values of $\bar{\epsilon}_{bm}$ and $\bar{\epsilon}_{am}$ for a specific b/a ratio and knowing the values of n, K, G, C, and α_z, the maximum pressure P_m can now be determined using equation (14). Fig. 4 is a plot of the maximum pressure P_m versus the b/a ratio for a material defined by the values of n, K, G, and C.

Conclusions

In this paper theory was developed for the determination of the plastic pressure-deformation relation in a thick-walled, closed-end cylindrical pressure vessel subjected to internal pressure. In order to present the complete solution the directional properties of the material were taken into account. In the theory developed large or finite strains are considered and a closed solution is found for the pressure-strain relation based on a modified log-log tensile stress-strain relation.

Theory was also developed for predicting the maximum pressure which a thick-walled, closed-end cylindrical pressure vessel can withstand. In working with the equations derived the material constants n, K, G, and C are encountered. The values of these parameters are determined using the data obtained from simple tension tests. Knowing the values of n, K, G, and C, the values of $\bar{\epsilon}_{bm}$ and $\bar{\epsilon}_{am}$ can be determined for any given wall ratio b/a. With these known values of $\bar{\epsilon}_{bm}$ and $\bar{\epsilon}_{am}$ for a specific b/a ratio and knowing the values of n, K, G, and C, the maximum pressure P_m can be determined using equation (14).

Acknowledgments

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DISCUSSION

S. M. Jorgensen[6]

This paper is the fourth and apparently last of a series dealing with the development of a theory applicable to thick-walled cylinders, including pressure tests on small scale cylinders to test the accuracy of the theory. Engineers concerned with the design of high pressure cylinders will feel deeply indebted to the authors for this very comprehensive and timely work. Anybody actively engaged in this kind of work will have encountered the confusion and almost chaotic ideas of both users and fabricators with regard to stress distribution in thick-walled cylinders and their ultimate strength. The authors have made a valuable contribution toward the establishment of safe and reliable methods of engineering of such equipment.

The experimental work suffers from one weakness: It does not include any of the high tensile materials in the 95,000 to 115,000 psi ultimate stress range, which is the material used almost exclusively with pressures of any magnitude, say in the 6000 to 15,000 psi pressure range. This omission is regrettable, because these steels are generally quite erratic, and therefor present more of a challenge for the theorist, but the writer readily agrees that an erratic steel is not exactly the best material for proving a theory.

The writer has worked on the same subject on and off for a number of years, which resulted in a theory [1][7] based upon the true tensile stress-strain properties and considering finite strains. In this respect it is very similar to the authors' theory, and differs mainly in using a numerical integration method to find the pressure as a function of the expansion.

The method was tested on four different high tensile steels [2] and showed excellent agreement between theory and tests.

There is another difference in approach between the two methods, which is rather interesting. The authors introduce the significant strain $\epsilon = 2/\sqrt{3}\cdot\epsilon_\phi$ into the expression for the pressure-summation or radial-pressure differential across a cylinder wall, which eliminates the factor $2/\sqrt{3}$ from their final pressure-summation equation. Apparently the factor is then re-introduced by using the stress-strain relationship $\sigma = f(\epsilon) = f(2/\sqrt{3}\cdot\epsilon_\phi)$.

In the method proposed by the writer, σ is taken as the stress corresponding to the strained value of $\epsilon = u/r$, and the summation is multiplied by $2/\sqrt{3}$. The two methods result in the same value for the internal pressure. Fig. 5 through Fig. 8 show the maximum internal pressure versus wall ratio relation for the authors' test vessels, evaluated on the basis of the method described by the writer.

Fig. 5 and Fig. 7 show two curves based on the scatter of points on the tensile stress-strain curves. The curves coincide with the authors' curves for exact isotropic and anisotropic theories. Fig. 6 shows only one curve as the scales of the tensile stress-strain curves were too small for a clear separation. For Fig. 8, the authors showed only one theoretical curve.

As will be seen by comparison with the authors' previously published test results, the two methods give identical results.

The tests of A-212-B material are the least satisfactory. The writer's tests [2] showed excellent agreement between tests and theory. This might be due to the method used in obtaining the stress-strain curve. The authors' nominal and true stress-strain diagrams presumably are the "average tension" types ordinarily used. In reference [2], at large strains, the true stress at the neck was determined using the analysis due to Bridgman [3], projecting an enlarged image of the neck on a screen and measuring the radii of the neck and the neck profile. The radii are used in a formula for a "correction factor" by which the average tension is to be multiplied to obtain the flow stress. Since the correction factor is less than unity, the strain hardening curve will rise less rapidly when flow stress rather than average tension is plotted against strain. The effect is to lower the theoretical maximum pressure when plotted against the wall diameter ratio. The effect is very pronounced for high tensile steels, but diminishes rapidly with low tensile, ductile steels.

The method has been invaluable in obtaining very close agreement between test points and theory of the proposed ultimate strength theory.

The writer doubts that it will be worth the added labor and efforts to distinguish between isotropic and anisotropic material in thick-walled vessel work. It is gratifying to have a method

[6] Foster Wheeler Corporation, New York, N. Y.

[7] Numbers in brackets designate References at end of this discussion.

that can be used in cases where a definite directional anisotropy can be established. This is possible only for the more ductile materials, and for moderate thicknesses. For thick plates and particularly for high tensile materials, the variation in material properties through the thickness direction makes it impossible to establish a definite directional anisotropy. And these are the materials one has to contend with in high pressure design.

From personal experience, the writer can vouch for the importance of the authors' work but wishes to emphasize that it cannot

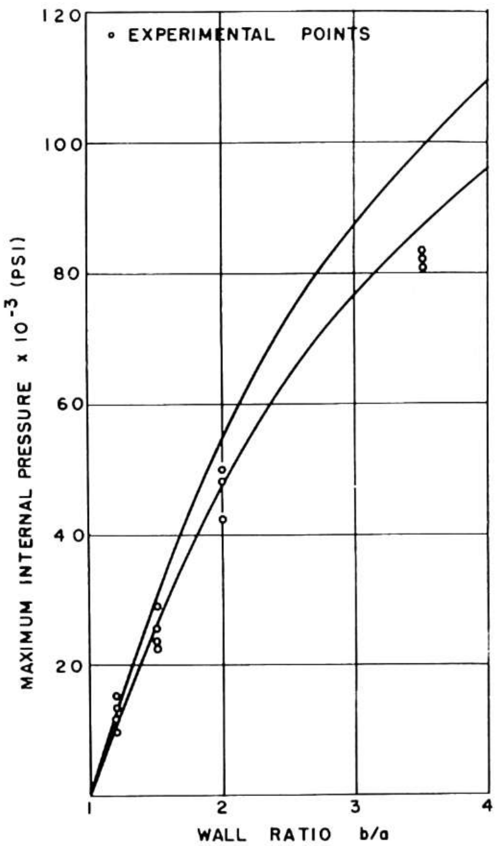

Fig. 5 A-212 Gr. B firebox—maximum internal pressure versus wall ratio

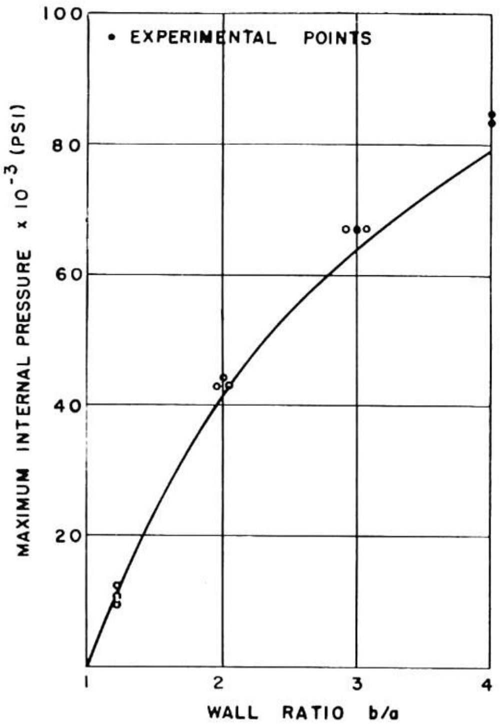

Fig. 6 A-285 Gr. C firebox—maximum internal pressure versus wall ratio

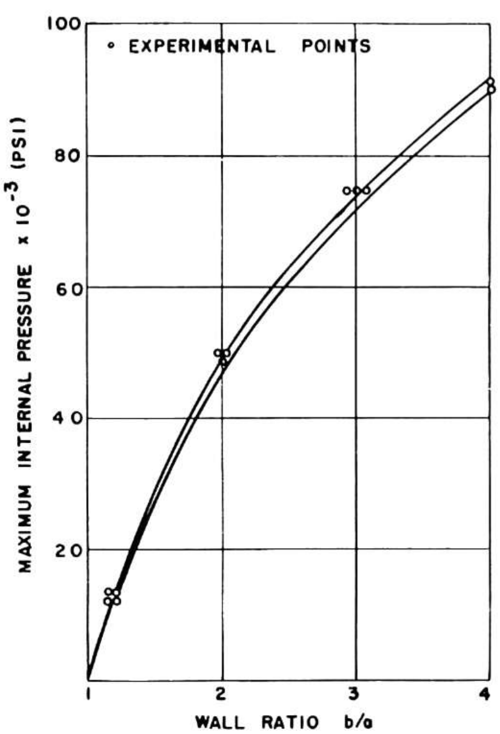

Fig. 7 T-304 stainless—maximum internal pressure versus wall ratio

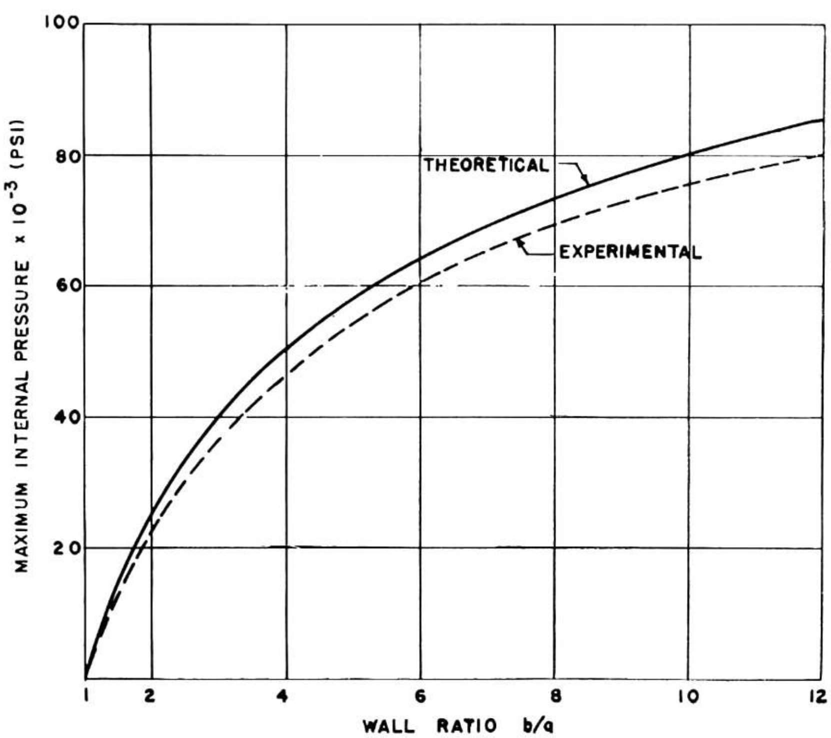

Fig. 8 6061-T4 aluminum—maximum internal pressure versus wall ratio be considered conclusive due to the limited range of materials used. The lack of authoritative design rules for thick-walled pressure vessels is a serious handicap which industry can ill afford, and it must be sincerely hoped that means will be found to continue this work.

References

1 S. M. Jorgensen, "Overstrain and Bursting Strength of Thick-Walled Cylinders," TRANS. ASME, vol. 80, 1958, pp. 561–570.

2 S. M. Jorgensen, "Overstrain Tests on Thick-Walled Cylinders," TRANS. ASME, vol. 82, 1960, pp. 103–121.

3 P. W. Bridgman, "Studies in Large Plastic Flow and Fracture With Special Emphasis on the Effect of Hydrostatic Pressure," McGraw-Hill Publishing Company, Inc., New York, N. Y., 1952.

F. P. J. Rimrott[8]

Engineering materials, especially those used for tubing and pressure vessels, are known to exhibit anisotropy to various degrees. It is therefore gratifying to see that the problem of anisotropy in pressure vessels is being considered.

The discusser would like to raise a point in connection with the assumption that $\epsilon_z = 0$, which is made at the beginning of the derivations. The authors fail to make a clear statement regarding the type of loading to which the pressure vessel is subjected. In order to obtain some information about the type of loading present, let it be assumed, for simplicity, that in the stress-strain relation (11), $K = 0$ and thus

$$\bar{\sigma} = C \tag{15}$$

Substituting the value $(\sigma_t - \sigma_r)$ of equation (b) into equation (3), and using eq. (15), one obtains

$$\frac{d\sigma_r}{dr_1} = \frac{1}{r_1}\sqrt{\frac{\alpha_z}{G}}\, C \tag{16}$$

and thus

$$\sigma_r = -\sqrt{\frac{\alpha_z}{G}}\, C \cdot \ln \frac{b_1}{r_1} \tag{17}$$

$$\sigma_t = \sqrt{\frac{\alpha_z}{G}}\, C \cdot \left(1 - \ln \frac{b_1}{r_1}\right) \tag{18}$$

where b_1 is the deformed outer radius.

The longitudinal stress is obtained by employing equations (a), (18), and (7c) and becomes

$$\sigma_z = \sqrt{\frac{\alpha_z}{G}}\, C \left(\frac{\alpha_{zt}}{\alpha_z} - \ln \frac{b_1}{r_1}\right) \tag{19}$$

Let us now assume that we have a closed-end cylinder. Adding all longitudinal forces in the shell

$$2\pi \int_{a_1}^{b_1} \sigma_z r_1 dr_1 = \sqrt{\frac{\alpha_z}{G}}\, C \left(\frac{\pi(2\alpha_{zt} - \alpha_z)}{2\alpha_z}(b_1^2 - a_1^2) + \pi a_1^2 \ln \frac{b_1}{a_1}\right) \tag{20}$$

where a_1 is the deformed inner radius.

Considering that $\sigma_r = -p$ when $r_1 = a_1$, equation (17) becomes

$$p = \sqrt{\frac{\alpha_z}{G}}\, C \ln \frac{b_1}{a_1} \tag{21}$$

The longitudinal force produced by the internal pressure is

$$\pi a_1^2 p = \sqrt{\frac{\alpha_z}{G}}\, C\, \pi a_1^2 \ln \frac{b_1}{a_1} \tag{22}$$

Subtracting equation (22) from equation (20) there remains a force

$$F = \sqrt{\frac{\alpha_z}{G}}\, C\, \frac{\pi(2\alpha_{zt} - \alpha_z)}{2\alpha_z}(b_1^2 - a_1^2) \tag{23}$$

For the ordinary closed-end vessel subjected to internal pressure, the force F is zero. This can only be the case for $2\alpha_{zt} = \alpha_z$. Since the analysis does not lose its generality when $\alpha_z = 1$ is used (see under Procedure for Determining the Theoretical Maximum Pressure), we find

$$\alpha_z = 1 \tag{24}$$

$$\alpha_{zt} = {}^1/_2 \tag{25}$$

from equation (7c)

$$\alpha_{rz} = {}^1/_2 \tag{26}$$

from equation (a)

$$\sigma_z = {}^1/_2(\sigma_t + \sigma_r) \tag{27}$$

from equation (7b)

$$\alpha_r - \alpha_{tr} = {}^1/_2 \tag{28}$$

from equation (7a)

$$\alpha_t - \alpha_{tr} = {}^1/_2 \tag{29}$$

Subtracting (29) from (28), we obtain

$$\alpha_t = \alpha_r \tag{30}$$

It can thus be concluded that for the stress-strain relation $\bar{\sigma} = C$, the material must be orthotropic with $\alpha_z \neq \alpha_t = \alpha_r$. If general anisotropy $\alpha_z \neq \alpha_t \neq \alpha_r \neq \alpha_z$ is considered, the condition $\epsilon_z = 0$ can only be fulfilled if an external longitudinal force

$$F = \sqrt{\frac{\alpha_z}{G}}\, C \cdot \frac{\pi(2\alpha_{zt} - \alpha_z)}{\alpha_z}(b_1^2 - a_1^2) \tag{23}$$

is applied, thus rendering the theory of limited value for actual pressure vessels.

A consideration of the longitudinal equilibrium of a closed-end vessel made of a material obeying the more general stress-strain relation (11)

$$\bar{\sigma} = K\bar{\epsilon}^n + C$$

will likely impose similar, if not more severe, restrictions upon the degree of anisotropy permissible if the assumption $\epsilon_z = 0$ is made and no external longitudinal force is to act.

Authors' Closure[9]

In the development of the theory proposed in our paper, the assumption was made that $\epsilon_z = 0$. This assumption was verified in subsequent tests on a number of cylindrical pressure vessels subjected to internal pressure only.

In the beginning of our paper it was stated that the purpose of the paper was to investigate the problem of finite plastic deformation of a thick-walled, closed-end stainless-steel pressure vessel subjected to internal pressure. The theory was developed with this in mind. The discusser wishes to assume, for simplicity, that in the stress-strain relation, $K = 0$. Such an assumption will definitely simplify the mathematics, but unfortunately for engineering materials K is never equal to zero. Therefore, to develop theory assuming $K = 0$ has academic interest but is of no practical significance.

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[9] By A. E. Dapprich.