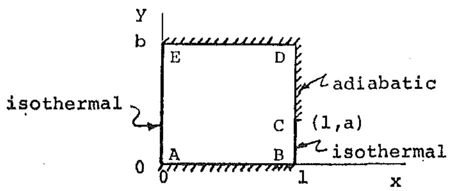

Fig. 1 Typical section

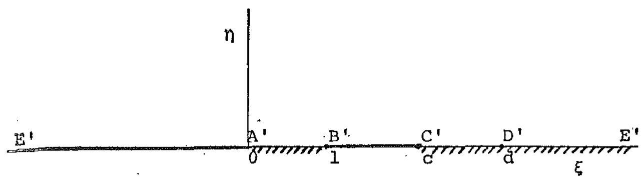

Fig. 2 Mapping onto upper half-plane

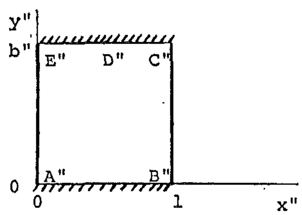

Fig. 3 Inverse mapping

The integral of equation (1) can be expressed in terms of elliptic integrals if the following transformation is used:

$$\tau = r^2$$

Then, for $\tau < 1$ and for $\tau = 0$ corresponding to $z = 0$,

$$z = \frac{2K}{\sqrt{d}} F\left[\frac{1}{\sqrt{d}}, \sin^{-1}(\sqrt{\tau})\right]$$

For $\tau > 1$, a second transformation is required:

$$d - r^2 = (d - 1)w^2$$

so that

$$z = \frac{2K}{\sqrt{d}} F\left(\frac{1}{\sqrt{d}}, \frac{\pi}{2}\right) - \frac{2K}{\sqrt{d}} i \left[F\left(\sqrt{\frac{d-1}{d}}, \sin^{-1}\left(\sqrt{\frac{d-\tau}{d-1}}\right)\right) - F\left(\sqrt{\frac{d-1}{d}}, \frac{\pi}{2}\right)\right]$$

For $z = 1$ to map onto $\tau = 1$ and for $z = 1 + ib$ to map onto $\tau = d$,

$$\frac{\sqrt{d}}{2K} = F\left(\frac{1}{\sqrt{d}}, \frac{\pi}{2}\right) \tag{3}$$

$$b = \frac{2K}{\sqrt{d}} F\left(\sqrt{\frac{d-1}{d}}, \frac{\pi}{2}\right) = F\left(\sqrt{\frac{d-1}{d}}, \frac{\pi}{2}\right) \Big/ F\left(\frac{1}{\sqrt{d}}, \frac{\pi}{2}\right) \tag{4}$$

The location of C' can be determined from the known value of $z(= 1 + ia)$ at C:

$$a = \left[F\left(\sqrt{\frac{d-1}{d}}, \frac{\pi}{2}\right) - F\left(\sqrt{\frac{d-1}{d}}, \sin^{-1}\sqrt{\frac{d-c}{d-1}}\right)\right] \Big/ F\left(\frac{1}{\sqrt{d}}, \frac{\pi}{2}\right) \tag{5}$$

The conductance of the strip heater as shown in Fig. 1 can be determined from the above transformation as follows:

1 The value of d is determined from equation (4) for the known value of b.

2 The value of c is determined from equation (5) for the known values of a and d.

3 The mapping in the $(\xi - \eta)$ plane, Fig. 2, is transformed via the same equations to the $(x'' - y'')$ plane shown in Fig. 3 to a problem with known conductance. The location of b'' is found from equation (4)

$$b'' = F\left(\sqrt{\frac{c-1}{c}}, \frac{\pi}{2}\right) \Big/ F\left(\frac{1}{\sqrt{c}}, \frac{\pi}{2}\right)$$

4 The conductance of Fig. 1 is calculated from [2]

$$\frac{Q}{k(\Delta T)} \equiv S = b''$$

It appears that the above solution is the only closed-form solution to the subject problem expressible in terms of tabulated functions.

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Taylor-Goertler Vortices and Their Effect on Heat Transfer[1]

Arun S. Mujumdar.[2] Although boundary layer development over curved surfaces is of great practical importance, to my knowledge there are no accurate prediction methods available which incorporate transverse and/or streamwise curvature effects. The effect of surface curvature is, of course, a special case of the more general effect of streamline curvature [12].[3] The curvature effects could be significant even on plane walls near separation. The problem dealt with by Professor McCormack and his co-workers is, therefore, a significant contribution to the existing literature, and will provide guidelines for future work in this area.

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[1] By P. D. McCormack, H. Welker, and M. Kelleher, published in the February, 1970, issue of the Journal of Heat Transfer, Trans. ASME, Series C, Vol. 92, No. 1, pp. 101–112.

[2] Mechanical Engineer, Research Division, Carrier Corp., Syracuse, N. Y.

[3] Numbers in brackets designate Additional References at end of Discussion.

The authors have shown that a linear analysis is not physically realistic. In fact, the nonlinear mechanism of vortex stretching seems to be crucial to all flows near walls; the viscous sublayer is perhaps a classical example [13]. A nonlinear analysis similar to that carried out by Sutera, Kestin, and Maeder would probably explain the increased heat transfer rates. The authors may be aware of the work of Williams [14] who solved the vortex-stretching model on an IBM computer using the versatile CSMP (Continuous System Modeling Program) language. The frequently awkward time and amplitude scaling procedures are not required in this digital equivalent of an analog computer. A major problem encountered in solving boundary-value problems numerically is that of searching the right initial conditions. I have successfully used a very elegant least-squares method proposed by Nachtsheim and Swigert [15] to obtain rapid convergence in the solution of boundary-layer-type equations. The technique is relatively insensitive to the initial guess.

Secondly, I wish to suggest that the 3 percent free-stream turbulence level could enhance the heat transfer rate appreciably if a streamwise pressure gradient exists [16]. Curiously enough, it appears that the physical mechanisms responsible for the transport phenomena are basically different depending on whether or not the flow contains oscillations of a discrete frequency (e.g., Taylor-Goertler vortices) or a wideband spectrum of frequencies (e.g., turbulence). Although only remotely analogous, one may consider the observations of Smith [17] with regard to the effect on heat transfer of the wake-induced periodicity near the forward stagnation point of a circular cylinder in a turbulence-free stream. His hot-wire data indicated that the wake induced periodic fluctuations (at Strouhal frequency) of about 12 percent of the mean velocity just outside the boundary layer. Suppression of these periodic oscillations by placing a splitter-plate in the near wake did not affect the heat transfer in the stagnation region (Re $\simeq$ 50,000). My own experiments at lower Reynolds numbers (5000–10,000), however, indicated a definite effect of these oscillations (up to 15 percent), the effect being more pronounced at lower free-stream turbulence levels and lower Reynolds numbers [18]. It is not intended to suggest or imply that the Taylor-Goertler vortices are similar to the stationary, three-dimensional vortex structures along the stagnation line induced by free-stream vorticity [19, 20]. The effects of the two, however, are curiously alike, and would certainly justify attempts to carry out a nonlinear analysis.

In this connection I would like to point out Karlsson's [21] analytical and experimental study of a flat-plate turbulent boundary layer when the free stream fluctuates sinusoidally about a constant mean. He showed that the periodic and mean motions become uncoupled (neglecting compressibilty) for small amplitude oscillations or higher amplitude oscillations at higher frequencies, i.e., nonlinearity may be neglected. In the case of the Taylor-Goertler vortices, the amplitudes and frequencies probably lie in the intermediate range where nonlinearity cannot be ignored.

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Authors' Closure

The contribution of Dr. Mujumdar is a very useful one. There appears to be a general consensus now that the non-linear mechanism of vortex stretching is necessary to explain the heat transfer effect found experimentally. The question of the similarity between Taylor-Goertler vortices and the stationary three-dimensional vortex structures along stagnation lines which occur in flow past cylinders and also past conical-shaped nose cones (in supersonic flow) is one that has yet to be resolved. There is one school of thought which considers that both effects are just pressure oscillations in the boundary layer. If this is so, then the periodic and mean motions should be decoupled. An experimental program to check this might be able to establish the difference between these two boundary-layer flows. In the Taylor-Goertler case, the periodic and mean motions are definitely coupled.

Hyperbolic Heat-Conduction Equation— A Solution for the Semi-Infinite Body Problem[1]

K. J. Baumeister[2] and T. D. Hamill.[3] In reference [1],[4] a step change in temperature was applied to a semi-infinite medium and the following equations and boundary conditions were considered:

$$\frac{1}{C^2}\frac{\partial^2 T}{\partial t^2} + \frac{1}{\alpha}\frac{\partial T}{\partial t} = \nabla^2 T \tag{1}$$

$$\frac{\alpha}{C^2}\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k\nabla \mathrm{T} \tag{2}$$

with the boundary conditions on temperature

$$T = T_0 \left.\frac{\partial T}{\partial t}\right|_{\substack{t=0\\x>0}} = 0 \quad \text{at } t = 0 \qquad x > 0 \tag{3}$$

$$T = T_w \quad \text{at} \quad z = 0 \qquad t > 0 \tag{4}$$

$$T \rightarrow T_0 \qquad x \rightarrow \infty \qquad t > 0 \tag{5}$$

In addition, the initial heat flux was assumed intuitively to be

$$q = 0 \qquad t = 0 \qquad x \geq 0 \tag{6}$$

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[1] By K. J. Baumeister and T. D. Hamill, published in the November, 1969, issue of the JOURNAL OF HEAT TRANSFER, TRANS. ASME, Series C, Vol. 91, No. 4, pp. 543–548.

[2] Lewis Research Center, Cleveland, Ohio.

[3] McDonnell Douglas, Huntington Beach, Calif.

[4] Numbers in brackets designate References at end of article under discussion.