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DISCUSSION

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The solutions presented in this paper are determined for a particular set of problems, within a general class for transient conduction in a finite slab, where the heat transferred to the slab is continuously positive or negative as time increases. As in many other studies of transient conduction, it is assumed that the temperature through the slab is initially uniform and that the slab has one face which is adiabatic. The process of transient conduction under these conditions can be considered as composed of two parts: an initial penetration, where the temperature distribution is very close to that found in a semi-infinite slab subject to the same surface conditions and thermal properties; then, once the temperatures at the adiabatic surface of the slab begin to change significantly, it becomes necessary to take account of finite thickness of the slab, with the temperature distribution at the end of the first part of the process being taken as the initial condition of the second part of the transient conduction process. In practice a sharp distinction between these two parts of the transient conduction process cannot be drawn. However, there exist certain approximate methods where this viewpoint offers a convenient framework for analysis. It appears that the authors are not aware of some of the work which has been done in this connection [18].[4] The results of these methods offer additional scope for comparison and interpretation of the solutions which have been presented. Some simplifications can be made in the relations which are used to estimate temperature distributions under conditions which lie outside the range of the charts. The approximate methods to be described also provide a means for estimation of temperature distributions through the thickness of a finite slab. In the calculation of thermal stresses and the determination of the thermal buckling behavior of plates and shells it is very desirable to have a good estimation of the temperature distribution *through* the thickness of the solid material involved. Problems of this nature have drawn some attention in the past, e.g. [19, 20], though in problems for which solutions have been presented the temperature distributions have been of a somewhat simple nature. However, the operating conditions of current and projected devices are such that greater attention needs to be given to problems such as that discussed in the present paper.

In the discussion which follows, it is found desirable to treat the problems of radiant heating and radiant cooling separately. However, some of the techniques which are involved apply to both heating and cooling. It should be understood, therefore, that some of the ideas used in connection with radiant cooling are discussed in connection with radiant heating.

The approximate method which is used in this discussion is based upon a variational method which has been extensively described by Biot [21, 22] who also provided some examples of its use. More recently, Lardner [18] and Richardson [23] have presented solutions to some further examples. As applied to the analysis of one-dimensional heat conduction, use of the variational method requires the assumption of a particular form of temperature profile and a sufficient number of generalized coordinates q are assigned to describe it. It is convenient (and often surprisingly accurate) to assume that the temperature profile is parabolic. Thus, for transient conduction in a semi-infinite body, it is assumed that the temperature profile is parabolic within the interval $0 \leq x \leq q_2$ and zero at all $x > q_2$. The surface temperature θ_0 is assigned to be the generalized coordinate q_1, so that the initially uniform slab temperature is taken as the reference zero. Then

$$\theta = T - T_i = q_1 \left\{1 - \frac{x}{q_2}\right\}^2; \; 0 \leqslant x \leqslant q_2. \qquad (31)$$

When it is necessary to describe the temperature profile within a finite slab, the temperature at the nonadiabatic surface is assigned to be the generalized coordinate q_1, again, while the temperature at the adiabatic surface is assigned to be the generalized coordinate q_3. The temperature distribution is then

$$\theta = T - T_i = (q_1 - q_3)\left\{1 - \frac{x}{L}\right\}^2 - q_3 \qquad (32)$$

In both cases the coordinate x is measured from the nonadiabatic surface. It may be noted that the parabolic profile assumed for the finite slab automatically satisfies the condition that the back surface is adiabatic, since the temperature gradient is zero there. For the problem which involves transient conduction in a semi-infinite body, the coordinate q_2 can be regarded as the penetration depth of the temperature transient. When unsteady conduction is occurring in a finite slab, at small times the penetration depth q_2 will be less than L. As q_2 increases with time, it will at some finite time become equal to L. Until that time it is possible to regard the transient as occurring in a semi-infinite body. The time required for q_2 to reach L is called the penetration time.

Radiant Heating

As shown by the authors, it is possible to use solutions for the temperature distribution in a slab of finite thickness subject to a constant heat flux into the slab as an approximation for transient radiant heating when the initial temperature ratio is near zero. Lardner shows that the solution of Carslaw and Jaeger is approximated well by the simpler expression

$$T_3 = T_i - \frac{F_o L}{k}\{N_{\text{Fo}} - 0.3328 - 0.8374 \exp(-21 N_{\text{Fo}})\} \qquad (33)$$

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[4] Numbers in brackets designate Additional References at end of discussion.

when N_{F_o} exceeds its penetration value of 0.149. Since constant heat flux has been assumed, it is also necessary to limit application of this equation to about $T_s/T_e < 0.4$. These limitations mean that the foregoing equation is limited to large values of N_{rh}. However, at very large N_{rh}, the temperature distribution in the slab as a function of time approaches rapidly to its asymptotic, analytic form, as given by the solution to equation (27).

It is worthwhile to note that a dimensionless time scale $\eta = N_{\mathrm{Fo}}/N_{rl}$ or N_{Fo}/N_{rc} arises naturally in the solutions of equation (27). If the transients in the charts given in the paper are replotted on this time scale, it is found that at large time the transients for different N_{rh} or N_{rc} tend to converge. For radiant heating, the convergence is virtually complete by the order of $\eta = 1.0$. At smaller η, it is observed that transients for small values of N_{rh} achieve a specific value of $(T - T_i)/(T_e - T_i)$ earlier than transients for large values of N_{rh}. Transients for finite values of N_{rh} appear in this sense to have a "lead-time" or "advance-time" over the asymptotic solution. It may also be noted that the variation of the dimensionless temperature $(T - T_i)/(T_e - T_i)$ as given by the asymptotic solution for the case $T_i/T_e = 0$ is very close to being a linear function of η over the range of dimensionless temperature from about 0.03 to 0.9. This covers the major range of variation of the slab temperature.

For a range of N_{Fo} the dimensionless temperature exceeds 0.4 before N_{Fo} exceeds its penetration value. Under these circumstances, and for cases when N_{rh} is large but there is interest in the temperature at small values of time, it becomes necessary to use an approximation based upon an analysis of transient conduction in a semi-infinite body. As mentioned before, an approximate solution for this is given by Lardner, and the dimensionless time at which penetration to the back of the slab occurs is given by

$$\eta = 0.149/N_{rh} \tag{34}$$

and the dimensionless temperature in the period up to this time is given by

$$\frac{T_s - T_i}{T_e - T_i} = 1.16\,\frac{(N)_{\mathrm{Fo}}{}^{1/2}}{N_{rh}}\left(\frac{T_e}{T_e - T_i}\right)\left\{1 - \left(\frac{T_i}{T_e}\right)^4\right\} \tag{35}$$

and

$$q_2 = 2.59(\alpha t)^{1/2} \tag{36}$$

The writer found that these expressions checked well with the charts given in the paper over the range of applicability of the equations. It may be noted that the range of values of N_{rh} for which these approximations can be applied right up to the penetration time is limited by the assumption of constant heat flux. However, within the range of N_{rh} for which these equations remain a good approximation, it is possible to exploit them in an approximate method for determining temperatures at larger times.

The manner in which the solution for transient conduction in a semi-infinite body can be used as a stepping stone for estimation of the temperature distribution at times greater than the penetration time for a finite slab can be explained as follows. Consider a slab at a time exceeding the penetration time. If the temperature at each point within the slab is increasing linearly with time, the temperature distribution at any instant of time within the slab will be parabolic. The surface temperature will determine the heat flux at any instant of time which crosses the outer surface into the slab. Provided that the rate of change of surface temperature with respect to dimensionless time remains constant, it would not be possible to tell the difference between the slab having a finite value of N_{rh} and an asymptotic slab from measurement of the rate of change of dimensionless surface temperature with respect to dimensionless time. This is a consequence of the fact that bodies of the same thickness and specific heat will require the same heat flux to raise the temperature at each point within the slab linearly with the time whether or not at any given time the temperature is uniform through the slab. For a given radiation configuration and initial temperatures, the temperature rise of the slab surface is proportional to the dimensionless penetration time η. The proportionality is greater than that associated with values of the asymptotic solution for large values of N_{rh}, and results in the surface temperature reaching a particular value at a time earlier on the dimensionless time scale than would the surface of an asymptotic slab. This time difference is the "lead-time" or "advance-time" noted earlier. These observations lead to the suggestion of a method of approximate calculation of temperatures in a slab at times greater than the penetration time as well as times less than this.

The approximation can be summarized as follows: For times less than the penetration time as given by equation (34) the temperature within the slab can be estimated from equations (35) and (36). When the dimensionless penetration time is achieved this time and the corresponding surface temperature should be calculated using the same equations. The dimensionless time at which the asymptotic solution for large N_{rh} achieves the same temperature should also be calculated. The difference between this latter time and the penetration time should then be determined; this time is termed the "lead-time." The surface temperature at any time subsequent to the penetration time can be estimated by assuming that it is the same temperature which would be achieved by the asymptotic solution at a time equal to the time of interest plus the lead-time. Conversely, if it desired to estimate the time at which the surface temperature will achieve a certain value, the time which would be taken by the asymptotic case to achieve that temperature can be calculated directly from the analytic solution, and then the lead-time subtracted from this. The temperature at the adiabatic surface of a slab can be assumed to lag behind that of the heated surface by the amount of the difference which is found when the penetration time is achieved. This approximation is restricted to values of the dimensionless temperature below 0.9, and to values of N_{rh} exceeding about 2.0.

To gain some idea of the order of accuracy of this approximation, a few sample cases were calculated and compared with the graphs presented in the paper. For the solutions where $T_i/T_e = 0$ the somewhat extreme example of $N_{rh} = 2.0$ gave a penetration time $\eta = 0.0785$ with a dimensionless temperature at the surface of 0.216; it was estimated that the dimensionless time required to reach this time by the asymptotic solution was $\eta = 0.225$. This gives an expected lead-time of 0.1465, while values which were measured from Fig. 4(c) of the paper gave estimates over a wide range of times for the lead-time of 0.142 to 0.150. As another example, the case of $T_i/T_e = 0.75$ with $N_{rh} = 5.0$ was taken; the dimensionless penetration time was estimated at $\eta = 0.030$ with the dimensionless surface temperature corresponding to this being 0.236. The estimated time for the asymptotic solution to achieve this temperature was $\eta = 0.094$, giving an expected lead-time of 0.064. The lead-time was estimated from Fig. 7(c) as about 0.065. It appears from a few sample cases examined on the basis of the charts that the temperature difference from the forward and back surfaces of the slab begins to decrease significantly once the slab temperature at the surface reaches a dimensionless value of about 0.8.

Beyond the bounds for reasonable use of the approximations described previously, there remains a range of the radiation number N_{rh} for which no very simple approximation can be provided. Small values of the radiation number (less than, say, 0.1) are encountered when the thermal conductivity is relatively small or when the slab thickness is relatively large. When the slab conductivity is small, it is unlikely that the boundary condition at the back surface of the slab is physically realistic; it would require the provision of a material having a thermal conductivity of a magnitude orders smaller than a quantity which is already small in terms of the practical range of thermal conductivities. On this basis it is to be expected that the penetration times for such slabs will be very long, and that the major change in surface temperature will occur over a time interval when the slab can be treated as a semi-infinite body. For such problems the solutions for the semi-infinite slab subject to a constant heat flux at the exposed surface cannot be used. Examination of the relationship between surface temperature and time as determined by the solutions presented in this paper for small values of the radiation number suggests that the surface temperature varies roughly as the nth power of the time over a fairly large interval in dimensionless temperature; it should be possible to base an approximate estimation on this fact.

The authors discuss the case where the radiation number is zero. They assume that this corresponds to the condition where a slab undergoes a step change in the surface temperature. Values are presented in the charts and comparisons are made with analysis in accordance with this assumption. It appears that the case where the radiation number is zero in fact presents an ambiguity. The case which the authors have treated is that where the numerator in the expression for the radiation number tends to zero. There is an alternative case where the radiation number becomes zero when the denominator within the expression for the radiation number becomes infinite. This latter case can occur, for example, when the slab thickness L becomes infinite. The solution to this latter problem then becomes formally identical with the solution for a semi-infinite slab. It may be noted that the dimensionless time scale $N_{\mathrm{Fo}}/(N_{rh})^2$ does not contain the slab thickness L explicitly, and it is possible for the radiation number to go to zero without this dimensionless time going to infinity. It is worthwhile to note that this dimensionless time scale is the one used by Lardner in discussion of transient conduction in a semi-infinite solid. It should be emphasized therefore that the case where the radiation number tends to zero does not necessarily mean that surface temperature of the slab undergoes a step change.

Radiation Cooling

Approximate methods of calculation of the transient temperature distribution in a finite slab subject to radiation cooling can be based upon the methods described in connection with radiation heating, but it is found that the range of utilityof each approximation is more limited.

Lardner [18] applied Biot's method to determine the transient temperature distribution in a semi-infinite slab which looses heat at a rate proportional to a power m of the surface temperature. Solutions presented include the case $m = 4$. This corresponds to the case where $T_e/T_i = 0$ which is treated in the paper here. As a starting point in application of this method, Lardner obtained two first-order simultaneous nonlinear differential equations; solutions were obtained by integrating the equations numerically by the Runge-Kutta technique, while the asymptotic solutions for short-times and for long-times were obtained analytically. The short-time solution begins to diverge significantly from the numerical solution once a dimensionless time $N_{\mathrm{Fo}}/(N_{rc})^2$ exceeds 0.01, and the long-time asymptotic solution converges very slowly with the numerical solution so that differences between them are still large when the dimensionless time has reached 100. Use of the conveniently simple asymptotic solutions for estimation of the temperature distributions at small times and for estimation of lead-times is therefore restricted in the range of dimensionless times over which each may be employed, in a way which does not occur for the corresponding solution used in connection with radiation heating. However, within the range of applicability it appears that the values obtained by Lardner compare well with the numerical solutions presented in this paper, and that estimates of lead-time can still be made in the same manner. With radiation cooling, it does appear that the real lead-time varies somewhat with time to a degree which is greater than that found with radiation heating. This is very probably due to the fact that the asymptotic solution itself is less linear with respect to time for radiation cooling than it is for radiation heating.

The short-time asymptotic solution for radiation cooling is identical in form with the solution for constant heat flux cited earlier in connection with radiant heating. Lardner showed that the numerical solution of the equations obtained by Biot's method agrees well with the solution of Abarbanel.

Lardner also discussed the case of a finite slab, with $T_e/T_i = 0$ and presented a solution for the surface temperature history for $N_{rc} = 1$. This appears to be in good agreement with the relevant curve in Fig. 8(c). The asymptotic solution for long times, valid for all N_{rc}, and obtained from application of Biot's method, is

$$\frac{T}{T_i} = (N_{rc}/3N_{\mathrm{Fo}})^{1/3}. \tag{37}$$

The dimensionless temperature in this equation is the complement of the dimensionless temperature used in Fig. 8 (to which the equation is appropriate). The writer observed when cross-plotting the curves in Fig. 8(c), as T/T_1 versus N_{Fo}/N_{rc} on log-log paper, that the dimensionless times taken to attain the asymptotic slope of $-1/3$ were indeed long: about $\eta = 10$ for $N_{rc} = 10$, and $\eta = 70$ for $N_{rc} = 0.1$. Agreement with the asymptotic solution was good beyond these times.

Much of the discussion on radiation cooling has centered so far on the case $T_e/T_i = 0$. When $T_e/T_i > 0$, it is of course found that dimensionless times required to achieve a particular dimensionless temperature decrease as T_e/T_i increases. Comparison with the charts given in the paper indicates that the lead-time method for estimating temperature histories is less accurate than with radiation heating. However, there does appear to be an approximate scheme by which dimensionless temperatures can be estimated for $T_e/T_i > 0$ if the temperature history for the same N_{rc} is known for $T_i/T_e = 0$. The approximation is at a specific value of dimensionless time $\eta = N_{\mathrm{Fo}}/N_{rc}$ and for a specific N_{rc}, the dimensionless temperature at $T_e/T_i > 0$ is greater than the dimensionless temperature at $T_e/T_i = 0$ by the factor f, given by

$$f = \exp\{1.44(T_e/T_i)\}. \tag{38}$$

The writer has not determined all the bounds within which this approximation is at all reasonable, but it is clear that it would not be used when either of the dimensionless temperatures involved is outside the range of about 0.1 to 0.9.

When the simpler methods of approximation fail, the effort required to make approximations based upon more involved methods begins to approach that required to obtain a numerical or analog solution. In some circumstances it is found that dimensionless surface temperatures are closely proportional to the nth power of time, and the iteration to find n in a specific case can converge very rapidly; for such cases the effort required exceeds only slightly that for the simpler methods.

The authors are to be thanked for providing the charts and associated details for this problem. Such charts are useful both for direct application and as a testing ground for approximate schemes.

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Authors' Closure

We wish to thank Professor P. D. Richardson for his informative comments. His presentation of an approximate method of solution should be of interest to all who are concerned with the analysis of transient nonlinear boundary value problems. The authors would like to emphasize that even though such approximate solutions may be restricted in their applicability, they do serve to complement analog solution charts and to verify their accuracy.

It is hoped that an insight will be gained through use of approximate techniques, both analog and analytical, which will eventually lead to exact analytical solutions, not only for the relatively simple model of this investigation, but also for nonlinear problems of greater complexity.