(9) is the upper limit of magnitude for particles heavier than the liquid.

The force acting on a gas bubble is greater than on a rigid particle, due to its own volume pulsation. This is clearly seen in Figs. 1–3, where the lines for bubbles and particles differ more and more with the increase of ω, A, or R_0, all of them being responsible for an increase in the bubble volume pulsations.

Attraction of pulsating bubbles has been known experimentally for many years [5] and this is a qualitative verification of the present theory. However, more theoretical and experimental work is needed in order that this attraction force can be of practical use. As it acts always in the same direction it may be particularly useful in separation processes, for example, to separate bubbles and particles from a liquid in a gravitational or nongravitational field. In addition, this force can be used to separate different kinds of particles, due to the fact that different densities cause different resultants between the "spherical attraction force" and gravity.

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DISCUSSION

R. Hickling[2]

The purpose of the paper is to demonstrate the existence of a spherical attractive force for bubbles and particles in the vicinity of a spherical oscillator. Such a general sweeping principle has not been demonstrated before and one wonders whether the authors are absolutely correct in concluding that it exists.

It is stated in the paper that the theory is based on the assumption that the particle or bubble stays essentially motionless in the liquid during a cycle of the oscillation. It is not clear how important this assumption is with respect to the theory developed in the paper, but it seems clear that the assumption itself is untenable.

At the low frequencies considered in the paper, about 20 Hz or less, a particle whose density is 2 relative to the liquid still moves essentially with the oscillation of the liquid, while the displacement of a bubble can significantly exceed that of the liquid (see, for example, Fig. 2 of the paper by Hickling and Wang[3]).

One can estimate what a typical liquid displacement will be, based on the example presented in the paper. For a spherical source with a radius of 50 mm, the radial displacement of the incompressible liquid at a distance of 100 mm from the center of the source will be 1/4 of that at the surface of the source. For the example presented in the paper, this would imply a total liquid displacement about its mean position of 5 mm. This is a significant magnitude in relation to the dimensions of most particles and bubbles of interest.

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[3]Hickling, R., and Wang, N. M., "Scattering of Sound by a Rigid Movable Sphere," *Journal of the Acoustical Society of America*, Vol. 39.2, 1966, pp. 276–279.

George Rudinger[4]

This interesting analysis demonstrates that a particle exposed to a radially oscillating pressure gradient experiences a net force in the direction toward the center of the oscillations. As indicated by the title of the paper, the fluid is considered inviscid, which is a legitimate assumption for the study of a particular aspect of the motion. However, before practical applications can be considered, some of which are pointed out by the authors, the relative importance of other forces must be evaluated, particularly viscous drag.

Purely oscillatory motion under the influence of viscous drag is of no interest in the context of the present study, but it should be established whether or not this force also produces a net drift velocity in addition to oscillatory motion. The answer to this question is not immediately apparent, but by way of illustration consider a particle released in a linearly oscillating flow described by

$$v = v_0 \sin(\omega t + \phi)$$

The equation of motion for a particle moving under the influence of Stokes drag then is

$$\ddot{x} = \frac{v_0 \sin(\omega t + \phi) - \dot{x}}{\tau} \tag{a}$$

where $\tau = 2R_p^2\rho_p/9\mu$ (μ is the viscosity of the fluid). The phase angle ϕ is introduced to allow the fluid velocity to be anywhere within its cycle at the instant of particle release. Integration of this equation for $x = 0$ and $\dot{x} = 0$ at $t = 0$ indicates oscillations with modified amplitude and phase with a superposed drift velocity which decays exponentially with the relaxation time τ. Integration of the drift velocity from zero to a time large compared with τ yields a mean particle position given by

$$\bar{x} = \frac{v_0\tau}{1 + (\omega\tau)^2}\,(\omega\tau \cos\phi - \sin\phi) \tag{b}$$

which can be positive or negative depending on the initial phase angle ϕ and has the extreme values $\pm\, \tau v_0\,[1 + (\omega\tau)^2]^{-1/2}$. Thus, particles released at random are spread over a band of twice this value.

Equation (a) also has been solved [1] for particle motion in the field of a standing wave

$$v = v_0 \cos(\pi x/2l)\sin\omega t$$

where $2l$ is the distance between nodes. This analysis indicates a drift velocity

$$v_D = \frac{v_0^2\pi(R_p/R_{popt})^2}{8l\omega\,[1 + (R_p/R_{popt})^4]}\sin\frac{\pi x}{l} \tag{c}$$

In this equation, R_{popt} represents the particle size for which the drift velocity becomes a maximum; it is given by

$$R_{popt} = \left(\frac{9\mu}{2\rho_p\omega}\right)^{1/2}$$

The drift velocity becomes negligible if the particle size is much smaller than R_{popt} because such particles can follow the gas motion without significant lag and if it is much larger, because such particles cannot follow the motion at all. A brief summary of this analysis is given also in reference [2].

Still other forces that affect the motion of a particle result from the added mass and from the history of the motion. The added mass for a spherical particle is one-half the mass of the displaced fluid. The history of the flow affects the motion, be-

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cause the flow field around an accelerating particle is different from that in steady flow (Basset force). These forces were analyzed by Hjelmfelt and Mockros [3], who investigated particle behavior in a turbulent fluid. They found important effects if the frequency of the oscillations is high enough, particularly in the case of gas bubbles, but indicated no drift velocity.

In view of these comments, it appears possible that a net particle motion could develop in a radially oscillating flow, and it would be interesting to know if the authors have contemplated this aspect. These remarks are not meant to detract from the analysis of the paper, but the questions raised must be answered before practical applications of radially oscillating flows can be considered.

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Authors' Closure

In this paper we tried to show by theoretical analysis the existence and the order of magnitude of the "spherical attraction force." It is obvious that experimental data regarding this effect are necessary.

In his discussion, Rudinger raises some interesting possibilities for evaluating viscous drag force effects. We have considered some of these effects in analyzing the behavior of gas bubbles in vertically vibrating liquids (in press, *Canadian Journal of Chemical Engineering*). In addition, the retardation of settling of particles [3] results essentially from viscous drag effects.

Regarding the comments of Hickling, we stated in the paper that for solid particles the use of equation (9), with the assumption that the particles stay motionless in the liquid during a cycle of oscillation, yields an upper limit for the force acting on particles heavier than the liquid. Therefore, the actual force acting on a particle of density 2.0 is expected to be smaller than the calculated value. For bubbles, the volume pulsations are a very important factor, as demonstrated by Figs. 1–3. Volume pulsations are possibly the predominant effect, as was demonstrated theoretically and experimentally even in uniformly oscillating liquids [1, 2].