$$= \mathcal{H}[-f(0)F_1(x, t, \mathcal{H}) - \int_0^\infty F_1 f'(x')dx'], \text{ etc.} \quad [64]$$

or

$$I_2 = -\mathcal{H}\left[\sum_{r=0}^{r=n} f^r(0)F_{r+1}(x, t, \mathcal{H}) + \int_0^\infty f^{n+1}(x')F_{n+1}dx'\right] \quad [65]$$

Evaluation of I_3. The third integral also can be expressed simply in terms of the F-functions. Thus

$$I_3 = \kappa\mathcal{H}\int_0^t F_{-1}(x, t - \tau, \mathcal{H})\phi(\tau)d\tau \quad [66]$$

Or, with the use of Equation [51], one obtains

$$I_3 = \mathcal{H}\int_0^t \frac{\partial F_1}{\partial t}\phi(\tau)d\tau = -\mathcal{H}\int_0^t \frac{\partial F_1}{\partial \tau}\phi(\tau)f\tau \quad [67]$$

$$= \mathcal{H}\,{}_0\Big[^t -F_1\phi(\tau) + \mathcal{H}\int_0^t F_1\phi'(\tau)d\tau \quad [68]$$

$$= \mathcal{H}F_1(x, t, \mathcal{H})\phi(0) - \frac{\mathcal{H}}{\kappa}\int_0^t \frac{\partial F_2}{\partial \tau}\phi(\tau)d\tau \quad [69]$$

$$= \mathcal{H}[F_1\phi(0) + \frac{1}{\kappa}F_3\phi'(0)] + \frac{\mathcal{H}}{\kappa}\int_0^t F_3\,\phi''(\tau)d\tau \quad [70]$$

The general result is

$$I_3 = \mathcal{H}\sum_{r=0}^{n} F_{2r+1}(x, t, \mathcal{H})\frac{\phi^r(0)}{\kappa^r} - \frac{\mathcal{H}}{\kappa^n}\int_0^t F_{2n+1}\phi^{n+1}(\tau)d\tau \quad [71]$$

DETERMINATION OF THE TEMPERATURE CO-EFFICIENTS

The final result for $T(x, t)$ is, exclusive of error terms

$$T(x, t) = \frac{1}{2}\sum_{r=0}^{n} f^r(0)\,[2\sqrt{(\kappa t)}]^r G_r\left[\frac{x}{2\sqrt{(\kappa t)}}\right]$$
$$-\mathcal{H}\sum_{r=0}^{n} f^r(0)\,F_{r+1}(x, t, \mathcal{H}) + \mathcal{H}\sum_{r=0}^{n}\frac{\phi^r(0)}{\kappa^r}F_{2r+1}(x, t, \mathcal{H}). \quad [72]$$

The derivatives appearing in Equation [72] can be expressed in terms of finite differences.[7] If, at time zero, the space distribution of temperature can be expressed by a second-degree polynomial in x, and the ambient temperature as a linear function of t, the following expressions apply to the various derivatives

$$f^0(0) = T_0{}^+ \quad [73]$$

$$f^1(0) = (4T_1 - 3T_0{}^+ - T_2)(\Delta x)^{-1} \quad [74]$$

$$f^2(0) = (T_0{}^+ - 2T_1 + T_2)(\Delta x)^{-2} \quad [75]$$

$$\phi^0(0) = T_a(+0) \quad [76]$$

$$\phi^1(0) = \{T_a(\Delta t - 0) - T_a(+0)\}(\Delta t)^{-1} \quad [77]$$

These expressions are used in this paper, although the result contained in Equation [72] applies to polynomials of arbitrarily high degree.

When the finite-difference Expressions [73–77] are substituted into Equation [72], the coefficients of the various equally spaced temperatures can be assembled. For the case where $t = \Delta t$ and $x = j(\Delta x)$ these coefficients are given in Equations [28–32] of the text. In presenting these coefficients, it is convenient to use the dimensionless sequence of functions defined by

$$F_n{}^* = F_n/(\Delta x)^n \quad [78]$$

[7] "Numerical Calculus," by W. E. Milne, Princeton University Press, Princeton, N. J., 1949.

Discussion

G. M. DUSINBERRE.[8] The author's analysis is to be compared on the one hand with suggested techniques which require the use of complicated formulas at all points in the system, and on the other hand with proposals which offer only a minor improvement for considerable extra work. In removing an awkward restriction at the boundary points which are usually in a minority, and this at a negligible cost in complication, the present paper is a valuable contribution to the calculation of the large systems which are of real practical importance.

A useful addition to the paper would be tables of coefficients for moduli 4 and 6, for two and three-dimensional systems.

C. M. FOWLER.[9] The author has made a useful contribution to the field of numerical analysis. Many of us are familiar with the difficulty he has surmounted, that of rapidly changing temperatures near a boundary. Further, it seems likely that the approach he has employed is capable of extension to other linear systems —vibrations resulting from sudden loading, for example.

It occurred to this reviewer that the author's boundary treatment might also be applied with advantage to nonlinear heat conduction. There is no rigorous justification for such an extension, of course, since the superposition principle is not valid for such systems. Nevertheless, as the following example shows, real gains are obtained using his boundary treatment instead of the conventional treatment, even though the system is nonlinear.

In this example, some boundary temperatures are calculated using both the author's and the more conventional treatment of the boundary. The author numbers these formulas [26] and [39] in his paper. The calculations are then compared to the differential-equation solution.

After a few attempts, a simple solution to the nonlinear conduction equation was found in which the thermal conductivity varied inversely as the square root of the temperature.

With the short nomenclature table below, it is easily verified that the conduction equation reduces to Equation [79], and that a simple solution is given by Equation [80].

Nomenclature

$k(T) = aT^{-1/2}$, thermal conductivity
c = const, heat capacity
ρ = const, density
$\alpha = 2a/c\rho$, constant
$h(T) = HT^{-1/2}$, heat-transfer coefficient
T_A = ambient temperature
b, d = arbitrary integration constants

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T^{1/2}}{\partial x^2} \quad [79]$$

$$T(x, t) = 9\alpha^2(t + d)^2/(x + b)^4 \quad [80]$$

Upon differentiating Equation [80], it is found that the boundary condition for simple convective cooling (Elrod's Equation [25]) is satisfied, provided the heat-transfer coefficient varies directly with $k(T)$, and that the ambient temperature is given by

$$T_A(t) = 9\alpha^2(t + d)^2(1 + 4k/bh)/b^4 \quad [81]$$

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[9] Physics Department, Kansas State College, Manhattan, Kans.

Numerical values selected for this study were: $a = 1.6$, $H = 3.2$, $C = 0.1$, $\rho = 8$, $b = 1$, $d = 1$. The differential equation solution is then given by Equation [82] while initial and boundary conditions are given by [83] and [84]. These last two equations furnish the data for the numerical computations

$$T(x, t) = 144(t + 1)^2/(x + 1)^4 \qquad [82]$$

$$T(x, 0) = 144/(x + 1)^4 \qquad [83]$$

$$\left(\frac{\partial T}{\partial x}\right)_0 = \frac{h}{k}(T_0 - T_A); \quad T_A = 432\,(t + 1)^2 \qquad [84]$$

In both numerical treatments of the boundary, Δx was taken as 0.2, and thermal conductivities were evaluated at the average temperature between $x = 0$ and $x = 0.2$. Nusselt numbers were calculated with these conductivities, but with heat-transfer coefficients evaluated at $x = 0$. A time interval $\Delta t = 0.05$ was chosen for the conventional treatment, thus fixing the modulus, which incidentally satisfied Dusinberre's stability criteria. In Elrod's treatment, the modulus M was set at 2.00, to take advantage of his Table 2. This choice established the time intervals. The quantities A_0 to E_0 necessary in applying Elrod's method were obtained by interpolation with the appropriate Nusselt number from this table.

The results of all these calculations are presented in the following table:

Time	T_A/x	0	0.2	0.4
0	432.0	144.0	69.4	37.5
0.05	476.3	158.8 (155.9)C	76.6 (75.5)C	
0.10	522.7	174.2 (170.0)C		
0.103		175.3 (177.4)E		

( )C.... Conventional
( )E........ Elrod

The single calculation using Elrod's method yielded a temperature of 177.4 deg at a time of 0.103. This compares favorably with the analytical value of 175.3 deg.

Three iterations, including one interior calculation, were necessary to obtain the temperature by conventional methods at approximately the same time. Thus at a time of 0.100, a temperature of 170.0 deg was obtained. This is to be compared with the analytical value of 174.2 deg.

The problem analyzed is admittedly rather ridiculous, a situation brought about by the necessity of having an analytical solution for comparison, but it does offer a rather severe test. The ambient temperatures selected furnish relatively large discontinuities at the boundary. Further, the thermal conductivity varies quite markedly from the boundary to the first interior point. Finally, the solution, which diverges to infinity with time, changes rapidly at the boundary. The author's treatment still appears to yield usable results.

Author's Closure

The author appreciates the comments of Professor Dusinberre and the ingenious illustration of a nonlinear application furnished by Professor Fowler. He would like to add here in closure a formula to which Equation [26] reduces when the internal temperature distribution is linear and the ambient temperature is constant

$$T(0, \Delta t) = NF_1^*T_a + \left\{1 - (N + 1)F_1^*\right\}T_0^+ + F_1^*T_1$$

where

$$F_1^* = \frac{1}{N}\left(1 - \epsilon^{\frac{N^2}{M}} \operatorname{erfc} \frac{N}{\sqrt{M}}\right)$$

Although somewhat less accurate than Equation [26], it can be readily used where tables are not available. The formula is stable for all N when $M \geq 2$ and is a suitable substitute for Equation [39].