Discussion

Allan H. Candee.[3] The writer will discuss Theorems 1 and 3 of the paper by referring to the kinematics of toothed gearing, inasmuch as the equivalent four-bar linkage for a pair of rotating gears behaves in the same way instantaneously as other four-bar linkages. Gears may operate at a constant ratio of angular velocities in which case their pitch curves are circular, or they may have noncircular pitch curves and a varying ratio.

In noncircular gears, at a position of maximum or minimum ratio, it is obvious that the pitch curves cross the line of centers at right angles, and that they then instantaneously conform to pitch circles. With these ideas in mind we can restate Theorem 1 in the terminology of gearing as follows:

At an extreme value of the velocity ratio in a pair of noncircular gears, in the instantaneously equivalent linkage, the line connecting the pitch point O to the instant center P of the common normal B-C of the tooth profiles is perpendicular to the normal itself.

This fact has long been known in the literature of kinematics and gearing. It is believed to have been discovered by Euler (1765) and it was used by Willis (1841) in a geometrical construction for locating centers of curvature in gear-tooth profiles.[4]

An easy way to arrive at the perpendicular relation described in the theorem can be shown by referring to Fig. 2 of the paper. Point O is the instant center (pole) of the relative motion between links A-B and D-C. At a maximum or minimum ratio of angular velocities point O must be instantaneously at rest in the stationary line A-D, which means that the point of link B-C at position O must be instantaneously moving in the direction of B-C. Because P is the instant center of B-C relative to line A-D, line P-O must therefore be perpendicular to B-C. The relation stated in the theorem has thus been known without benefit of mathematical analysis.

During the last Annual Meeting of the Society there was some discussion about accelerations in linkages. The writer has not had much occasion to deal with accelerations, but it occurred to him that the conditions in noncircular gearing could be used to derive an equation for the value of angular acceleration. This was done a short time later, and a brief outline of the method used may be interesting. Only a very simple diagram is needed, Fig. 13, herewith.

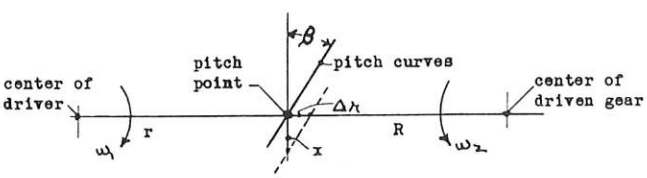

Fig. 13 Displacements in Noncircular Gears

For a small increment of time Δt, there is a displacement $\Delta x = r\omega_1\Delta t$ of the pitch curves, and a corresponding increment $\Delta r = \Delta x \tan \beta$ of the pitch radii r and R. Originally the angular velocity of the driven gear is $\omega_2 = (r/R)\omega_1$. After the time interval

$$\omega_3 = \frac{r + r}{R - r}\,\omega_1$$

The velocity increment of the gear is $\Delta\omega = \omega_3 - \omega_2$.

Making substitutions and dividing by Δt easily yields the angular acceleration of the gear

$$\alpha = \lim_{\Delta t = 0}\left(\frac{\Delta\omega}{\Delta t}\right)$$

or

$$\alpha = \frac{r(r + R)}{R^2}\tan\beta\,\omega_1{}^2$$

Stated in words, this means that

Angular acceleration of driven gear

$$= \frac{\text{(center distance)(radius of driver)}}{\text{(radius of driven)}^2}\tan\left(\begin{matrix}\text{angle of}\\ \text{pitch curves}\end{matrix}\right)\left(\begin{matrix}\text{angular velocity}\\ \text{of driver}\end{matrix}\right)^2$$

The equation derived as shown can be rewritten to conform exactly with Equation [16] of Theorem 3 of the paper. This requires recognizing (a) that angle β is the complement of angle λ of Figs. 5 and 12, (b) that the pitch point of the gears is located between the centers of rotation, and (c) that the velocity ratio $v = (r/R)$.

The writer had supposed that angular velocities in linkages were fully covered in kinematic literature, but he does not know whether the method of derivation shown in the foregoing is new or not. It seems to be shorter and simpler than the one presented in the paper.

V. L. Doughtie.[5] The author is to be commended for his contribution to the study of the four-bar mechanism. He gives a simple and direct method of obtaining the angular-velocity ratios of one link relative to another and the extreme values of these ratios. The design of a linkage for prescribed extreme values of the velocity ratio is presented. He also gives a method of obtaining the angular acceleration of the follower of a four-bar mechanism in which the driving crank rotates at a constant angular velocity. This tool is also applied to the slider-crank mechanism in which the crank rotates at a constant angular velocity. For various ratios of the length of crank to connecting rod, the linear velocity ratio of the sliding member to the crankpin, the maximum value of this ratio, and the position of the crank for the maximum value of the linear velocity of the slider are obtained. The acceleration of the slider is also obtained. The comments which follow are suggestions and not criticisms of the paper.

The terms "four-link mechanisms," "four-bar mechanisms," and "four-bar linkages" have the same meaning throughout the paper. Would it be desirable to standardize some of the terms we use in "Mechanism?"

The notation could be improved. Why use v for the angular velocity ratio? At one point v is used as "the angular velocity of the follower," and v_0 is used to represent the linear-velocity ratio. Referring to Fig. 1, O, O', O'' are upper case in the text but lower case in the vector diagram in Fig. 1. The letter O, in Fig. 5, is upper case throughout. In Fig. 5, a is used for the length of the crank but in Fig. 9, r is used for the length of the crank. Also, in Fig. 5, b is the length of the connecting rod whereas in Fig. 9, c is used. A consistent notation would cause the paper to be more easily understood.

It was hoped that the author's Theorem 1 would give a quick solution to the old problem of finding the maximum angular velocity of the follower crank of the nonparallel equal crank mechanism when the speed of the driving crank is known. Since the location of O when O-P is perpendicular to the connecting link B-C is not definite, the method cannot be used in obtaining an accurate value.

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[4] "A Brief Account of Modern Kinematics," by A. E. R. de Jonge, Trans. ASME, vol. 65, 1943, discussion on p. 680.

[5] Professor, Department of Mechanical Engineering, University of Texas, Austin, Texas. Fellow ASME.

In the determination of the angular acceleration of the follower (Theorem 3) the angle λ must be measured very accurately and, as stated, the method becomes unreliable when angle λ approaches zero.

The application to the slider-crank mechanism is interesting. The curves of Figs. 10 and 11 should be useful. The solution of Equations [22] to [26] would be rather laborious for simple problems.

Theorem 1 is certainly a valuable tool in analyzing the four-bar mechanism. Again, let me compliment the author.

N. Rosenauer.[6] The author is a notable representative of the family of kinematicians in the United States and is to be congratulated on this ingenious contribution to the science of kinematics. One could imagine that the last word in the investigation of the properties of the "collineation axis," established by Bobillier, has already been said, but the author has found an additional interesting property expressed in his Theorem I. Due tribute must also be paid to Prof. A. S. Hall for its proof given in the Appendix. All derivations in the paper are simple, straightforward and have, without any doubt, practical value since they are easily applicable.

The writer, personally, had the opportunity of checking some results of one of his latest papers No. 746 under the title "Application of Complex Variables to the Synthesis of Four-Bar Linkages With Prescribed Extreme Values of the Output Angular Velocities," which has been accepted for publication in the *Journal of Applied Mechanics*. The author's criterion (Theorem 1) works perfectly.

The derivation of the angular acceleration of the follower link as shown in Fig. 5, although not so simple, is very interesting, being connected with the direction of the collineation axis. This shows again the importance of the collineation axis and another practical application of it.

Fig. 7 shows a useful method for the determination of the direction of the collineation axis. In Fig. 5 it would be useful to repeat the notations of link $AB:a$ and of link $CD:c$ since these notations appear in the derivation of Theorem 3.

In connection with the slider-crank mechanism, the writer had again an opportunity to check some results obtained in a paper under the title "Application of Complex Variables to the Synthesis of Quick-Return Slider-Crank Mechanisms," which he is preparing for publication. Again, the author's criterion works perfectly, the only inconvenience in the case is that the collineation axis falls completely outside the drawing.

Finishing the discussion, the writer suggests the author apply his theory to the solution of the problem of designing a four-bar linkage with prescribed extreme values of the output angular velocity.

Author's Closure

The remarks of Mr. Candee concern the well-known relations between the kinematics of linkages and those of gearing. The type of derivation suitable for inclusion in a paper depends not only upon whether it is primarily geometrical, analytical, elementary, or advanced, but is influenced also by considerations pertaining to a unified and logical development and presentation. The derivation of Theorem 1 in Mr. Candee's discussion, as well as others, both geometrical and analytical, have been intuitively clear to the author and probably also to others at the time this investigation was conducted, which time ended in the spring of 1954. The works referred to in footnote 4 in Mr. Candee's discussion deal with gearing rather than with linkages. In connection with other investigations of velocities, the author would like to refer Mr. Candee to the introduction of this paper.

The derivation of Theorem 3 given by Mr. Candee is valuable and demonstrates clearly the connection between noncircular gearing and the equivalent four-bar linkage. In this connection, the author would like to state his admiration of, and respect for, Mr. Candee's expert knowledge of the kinematics of gearing and gear generation.

The comments of Professor Doughtie (whose text on mechanisms is well known), concerning nomenclature and notation, are well taken and appreciated. The terms "four-link mechanisms," "four-bar mechanisms," and "four-bar linkages" should indeed not have been used interchangeably. These are but one instance of terms in kinematics which are not properly defined, used, and understood. This of course in no way excuses their usage in this paper.

At the 1953 Annual Meeting of the Society a suggestion was made to standardize terminology in kinematics by Mr. A. E. R. de Jonge,[7] and it is believed that such standardization would form a suitable subject matter for an ASME-sponsored project.

It is true that the symbol v for angular-velocity ratio is not ideal. What other symbol would be better and would not lead to confusion with the symbol for the angular velocity of a crank? The author hopes to better his record in terminology and notation in future work, and regrets that the observations of Professor Doughtie and Dr. Rosenauer could not conveniently be incorporated into the paper at this late date.

The application of Theorem 1 to the nonparallel equal crank linkage does, it is true, lead to an indefinite position of O. Because of symmetry, however, it can be seen that as the links approach the in-line position, line O-P becomes vertical and symmetrical relative to the linkage, so that in the limit point O is halfway between B and D. This leads to a maximum value of v equal to $(AD + AB)/(AD - AB)$ as can be derived also from the treatment involving equivalent elliptical gearing.[8]

It was with considerable satisfaction that the author learned of the verification of some of the results by Dr. Rosenauer, one of the leading authorities in the field of kinematics. Some of Dr. Rosenauer's work in complex variables can now be studied in the English language[9] and it is to be hoped that the investigations concerning extreme values of the velocities will soon become available in the English language likewise. The author agrees with Dr. Rosenauer's comments regarding notation and intends to follow up Dr. Rosenauer's suggestion to apply the theory to the problem of designing a four-bar mechanism with prescribed extreme values of the output angular velocity.

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[7] "An Analytical Approach to the Design of Four-Link Mechanisms," by F. Freudenstein, Trans. ASME, vol. 76, 1954, p. 492.

[8] "Elements of Mechanisms," by V. L. Doughtie and W. H. James, John Wiley & Sons, Inc., New York, N. Y., 1954, pp. 124–125.

[9] "Complex Variable Method for Synthesis of Four-Bar Linkages," by N. Rosenauer, *Australian Journal of Applied Science*, vol. 5, December, 1954, pp. 305–308.