Integrate Equation [96] and impose conditions that when

$$\phi = 0 \text{ then } v = 0 \text{ and } \frac{dv}{d\phi} = 0$$

The displacement of the cross section at 0 is

$$v = \frac{MR^2}{C} [1 - \cos \phi] \sin \alpha + \frac{MR^2}{2} \left[\frac{1}{KEI} + \frac{1}{C} \right] [\cos \alpha \sin \phi - \phi \cos (\alpha - \phi)] \quad [97]$$

and the rotations at 0 are

$$\beta = \frac{MR}{C} [\sin \alpha - \sin (\alpha - \phi)] - \frac{v}{R} \quad [98]$$

$$\gamma = \frac{-MR}{C} \sin \alpha \sin \phi - \frac{MR}{2} \left[\frac{1}{KEI} + \frac{1}{C} \right] [\cos \alpha \cos \phi - \cos (\alpha - \phi) - \phi \sin (\alpha - \phi)] \quad [99]$$

By setting $\alpha = \phi = \frac{\pi}{2}$ in the solutions of v, β, and γ, the values are obtained for end deflections of quarter bends as applied to the Piping Handbook cases Nos. 8, 9, and 10.

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Discussion

WILLIAM HOVGAARD.[8] This paper brings the difficult problem of three-dimensional pipe bends one important step nearer to its complete solution.

The writer has given considerable study to this problem and in 1935 and 1937 read two papers[9,10] before this Society giving an algebraical solution based on the theory of deflection by bending.

In that solution account was taken of the effects of forces and couples acting normal to the plane of the bends, but in so doing the theory of the bending of solid curved rods was applied; the distorting effect on the tube section, with which this paper deals, was not included.

Much attention was given to the determination of the relative importance of the terms in the expressions for rotation and displacement caused by bending of the pipe bends out of their plane. These terms, being apparently small, were referred to as "secondary" terms.

The papers [9,10] were accompanied by a numerical calculation for a certain typical three-dimensional pipe system and, finally, elaborate calculations were carried out for the same system with various lengths of tangents, both for the case when the secondary terms were included and for the case when they were neglected. The results are presented in a table[11] giving the terminal reaction forces and couples.

It was found from this and several other calculations that, in such pipe systems as ordinarily occur in the propulsive machinery of destroyers, the secondary terms for any length of tangents were so small that they could be neglected.

Still the writer felt some doubt of the completeness of the solution and in the concluding remarks to the discussion on his paper,[10] he made the following statements:[12]

"The algebraic method is straightforward, and the only point which needs to be cleared up is the behavior of curved pipes when subject to forces and couples acting normal to their plane. . . . the few tests which have been made so far on this problem do not seem to check with the formulas. It is desirable that more complete tests be made, ..."

It is fortunate that this problem has now been taken up by the author, who has shown that the flexibility of bends perpendicular to their plane is greater than expected from the application of the "rod" theory, on account of distortion of the cross action of the pipe, not hitherto taken into account.

The solution presented is very complete from the mathematical point of view, and is corroborated by some preliminary experiments. It does not seem quite clear, however, what the influence is quantitatively of the deformation of the pipe section pointed out by the author.

It would be of great interest if a calculation in accordance with the author's theory could be made for the pipe bend used in the numerical example of the writer's papers,[9,10] in which case the results could be compared with those given in the table referred to.[11]

The value of such a theoretical study would be much enhanced by an experimental test of the same pipe bend, the complete dimensions of which are given in the writer's first paper.[9]

Unfortunately, the writer has not the time or facilities available for undertaking such work himself, but it is suggested as a suitable thesis for students in a technical university.

ARTHUR MCCUTCHAN.[13] The author is to be commended both for his intelligent approach to the problem and for the realistic

[8] Professor Emeritus, Department of Naval Construction, Massachusetts Institute of Technology, Cambridge, Mass. Mem. A.S.M.E.

[9] "Stresses in Three-Dimensional Pipe Bends," by William Hovgaard, Trans. A.S.M.E., vol. 57, 1935, pp. 401–415; discussion, Trans. A.S.M.E., vol. 58, 1936, pp. 391–400.

[10] "Further Studies of Three-Dimensional Pipe Bends," by William Hovgaard, Trans. A.S.M.E., vol. 59, 1937, pp. 647–650; discussion, Trans. A.S.M.E., vol. 60, 1938, pp. 596–605.

[11] Bibliography reference (10), p. 650.

[12] "Further Studies of Three-Dimensional Pipe Bends," by William Hovgaard, author's closure to discussion, Trans. A.S.M.E., vol. 60, 1938, p. 605.

[13] Engineer, Engineering Division, The Detroit Edison Company, Detroit, Mich. Mem. A.S.M.E.

manner in which he has fitted his findings into the general theory of flexibility of pipe bends.

Ever since the tests described by Messrs. E. T. Cope and E. A. Wert[14] were made, the writer has sought a rational explanation for the greater flexibility of curved pipe acted on by forces and moments perpendicular to the plane of the bend. In discussing discrepancies between theory and results obtained on those tests, Dr. S. Timoshenko suggested that there must be some unknown "form factor" which would need to be applied to secure the desired agreement between test and theory. A test program was initiated by H. E. Mayrose under Dr. Timoshenko's direction which substantiated the previously observed fact that the deflections due to sidewise loading were greater than predicted by existing theory. The results of these tests were reported by Professor Mayrose in his discussion[15] of a paper by Prof. Hovgaard.[9] Although Professor Mayrose suggested that the K factor used to correct for flattening of the cross section in formulas for two-dimensional pipe bends might be applied also to three-dimensional problems, he did not observe, nor attempt to measure, distortion of the pipe cross section when subjecting bends to sidewise loading. The fact that the tubes which he tested were small, 1 in. OD and less, may have served to obscure the flattening of the cross section at 45 deg to the plane of the bend.

It would appear, therefore, that the author has made a real contribution to knowledge of the behavior of pipe bends through his discovery that flattening of the pipe cross section occurs under sidewise loading.

Unfortunately, incorporating the K factor in equations for sidewise loading further complicates the already involved problem of determining the flexibility of piping lying in three planes. In order to appraise the effect of including the K factor in the equations for sidewise loading, the illustrative three-dimensional problem,[16] given in the "Piping Handbook," was recalculated. The resultant of the end reactions was found to be reduced by less than 3 per cent, when the so-called secondary rotations were neglected. However, the rotations in the complementary planes assume much greater values when the K factor is applied so this comparison is not a fair measure of the differences to be expected. For $K = 0.5$, the rotation in the plane perpendicular to the free end in Case 9 becomes $^1/_7$ of the rotation in the plane tangent to the free end rather than $^1/_{12}$ as when $K = 1$. This secondary rotation increases with reduction in value of K, until for $K = 0.1$, the secondary rotation is approximately $^1/_2$ as great as the principal rotation. The same changes take place in Case 10 where torsion acts on the end of a quarter bend.

The relative magnitude of the rotations in planes perpendicular and tangent to the free end caused by a sidewise force, Case 8, is greatly affected by the value of K. For $K = 1$, the rotation in the plane perpendicular to the free end is approximately $^1/_2$ as great as in the tangent plane, while for $K = 0.1$, the relations are reversed and the rotation in the plane perpendicular to the free end is 1.3 times the rotation in the plane tangent to the free end.

In attempting to apply these changes to Cases 8, 9, and 10 (Piping Handbook), the writer has made the abscissas of the bending-moment diagrams in the graphoanalytical method equal to the algebraic sum of the coefficients of rotation in planes perpendicular and tangent to the free end rather than to base them on rotation in the principal plane of action as at present.

It would be helpful if the author would include an example illustrating the application of the revised cases to a three-dimensional problem. Two 60-in-radius arcs of 12-in nominal diam $^1/_2$-in-wall pipe, disposed in planes at right angles to each other with no connecting straight pipe, would appear to be a good test case, since it would seem that two of the reacting forces should be in exact balance.

In view of the results so far obtained by the writer there is some uncertainty as to just how the revised cases can be combined to best advantage to give a complete solution of a three-dimensional problem. If the author finds it possible to include a solution of such a three-dimensional problem in his closure it should prove most helpful.

[14] "Load-Deflection Relations for Large, Plain, Corrugated, and Creased Pipe Bends," by E. T. Cope and E. A. Wert, Trans. A.S.M.E., vol. 54, 1932, FSP-54-12, pp. 115–143.

[15] Discussion by H. E. Mayrose, of "Stresses in Three Dimensional Pipe Bends" by William Hovgaard, Trans. A.S.M.E., vol. 58, 1936, pp. 395–397.

[16] Author's Bibliography, reference (11), pp. 629–643.

D. B. Rossheim[17] and A. R. C. Markl.[18] This paper makes a valuable contribution to the understanding of thin curved tubes, and assumes particular significance in view of the excellent test data adduced in support of the theoretical development. That the flexibility and stress-intensification factors are identical for flexure in and transverse to the plane of curvature should not cause surprise; on the contrary, it is surprising that the fact has escaped attention until the present time. Fig. 22 of this discussion clearly establishes that transverse bending (diagram b) represents no more than an intermediate condition between the two extremes of bending in the plane illustrated by (a) and (c), which show, respectively, the distortion produced by moments tending to close the bend (increase curvature) and open the bend (decrease curvature).

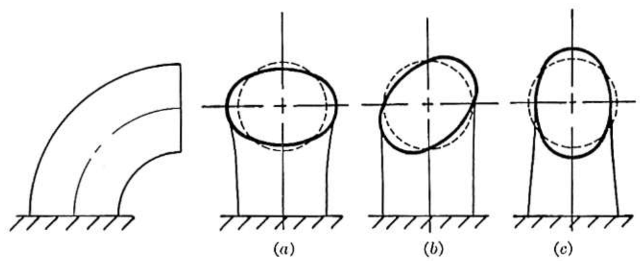

Fig. 22 Distortion in Pipe Bend

(Undistorted side view is shown at left of bend. a, Distortion of bend caused by moment tending to decrease radius of curvature of bend. b, Distortion of bend caused by moment acting perpendicular to plane of bend. c, Distortion of bend caused by moment tending to increase radius of curvature of bend.)

Taking advantage of the attention which those interested in expanding present-day knowledge of thermal-expansion effects in pipe will give to this paper, and without detracting in any way from the importance of the author's contribution to this subject, we feel it opportune to raise some of the most pressing unsolved problems, in the hope that further thought and study will be stimulated thereby:

1 *Limitation of Formulas.* Two approaches are available for the calculation of toroidal shapes, the thin-tube theory, which the author expands in this paper, and the curved-bar theory. Neither is complete, the former neglecting the displacement of the neutral axis toward the center of curvature, and the latter disregarding cross-sectional distortion. The one is nearer to the truth for thin tubes and the other probably more suited to sharp-radius thick-walled elbows. A definition of the limits of each is needed, or preferably a new approach combining all significant effects.

2 *Modifications Due to Pressure.* The thin-tube theory considers an external bending moment only; in actual applications, either internal or external pressure and some axial loading and shear are involved in addition to bending. Since the increased

[17] Consulting Engineer, M. W. Kellogg Company, New York, N. Y. Mem. A.S.M.E.

[18] M. W. Kellogg Company, New York, N. Y.

flexibility of curved pipe is due to its ovalization under flexure, which would be opposed by internal pressure, the question arises whether and to what extent the ovalization persists under internal pressure; also, whether it is accentuated by external pressure.

3 *Stability Considerations.* The distortion and stress redistribution, attendant to bending of a curved pipe, can be expected to affect its collapse resistance, both under external pressure and due to sustained axial forces or moments.

4 *Significance of Calculated Stresses.* Assuming that the stress-intensification factors can be properly evaluated, their significance with regard to ultimate failure still has to be established. Stresses are localized and, depending upon the capacity of other parts of the system to absorb overload, they should or should not be compared with normal design stresses. For steady stress, Hovgaard[19] has expressed the opinion that a design to the elastic limit will be safe. For cyclic conditions, Dennison[20] and the writers[21] have made tests providing a rough indication of the limits of safety. In this connection, it is noted that fatigue tests on bars do not even truly predict the endurance of straight pipe; also, that there is evidence that straight pipe has somewhat greater flexibility than indicated by the bar theory, a phenomenon which is as yet unexplained.

5 *Other Allied Problems.* Without intention to discourage, it is noted that there are numerous other effects pertinent to the subject of flexibility of piping, such as those in miter bends, at tees or laterals, in flanges, etc., about which practically nothing is known and which are, for this reason, often disregarded entirely by the uninitiated. Even a crude evaluation of these effects would do much to advance the basis for a proper piping-design practice.

[19] "Further Research on Pipe Bends," author's Bibliography (2), p. 292.

[20] "The Strength and Flexibility of Corrugated and Creased Bend Piping," *Journal*, American Society of Naval Engineers, vol. 47, 1935, pp. 343–432.

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Author's Closure

The practical engineer's first question, when a refinement of theory is introduced, is how greatly will the new theory affect the results as formerly derived. The answer to this question can only be given with exactness for specific cases. A general answer must necessarily be qualified. For many piping arrangements the contributions of transverse bending of bends play but a small part in determining the flexibility of the system. In such cases large changes of the transverse flexibility of the bends will have little effect on the flexibility of the system. This is the case with most piping systems.

Mr. McCutchan has found that for the illustrative three-dimensional problem given in the "Piping Handbook," the resultant of the end reactions is changed by less than 3 per cent by the introduction of the transverse-rigidity factor. The author has calculated the same problem by the method of Karelitz and Marchant,[22] in which secondary rotations have been included, and has found the resultant end reactions to be changed by less than 3 per cent. Evidently in cases of this type neither secondary rotations nor flexibility changes of transverse bending greatly affect the flexibility of the system. Piping systems that

[22] Bibliography (9).

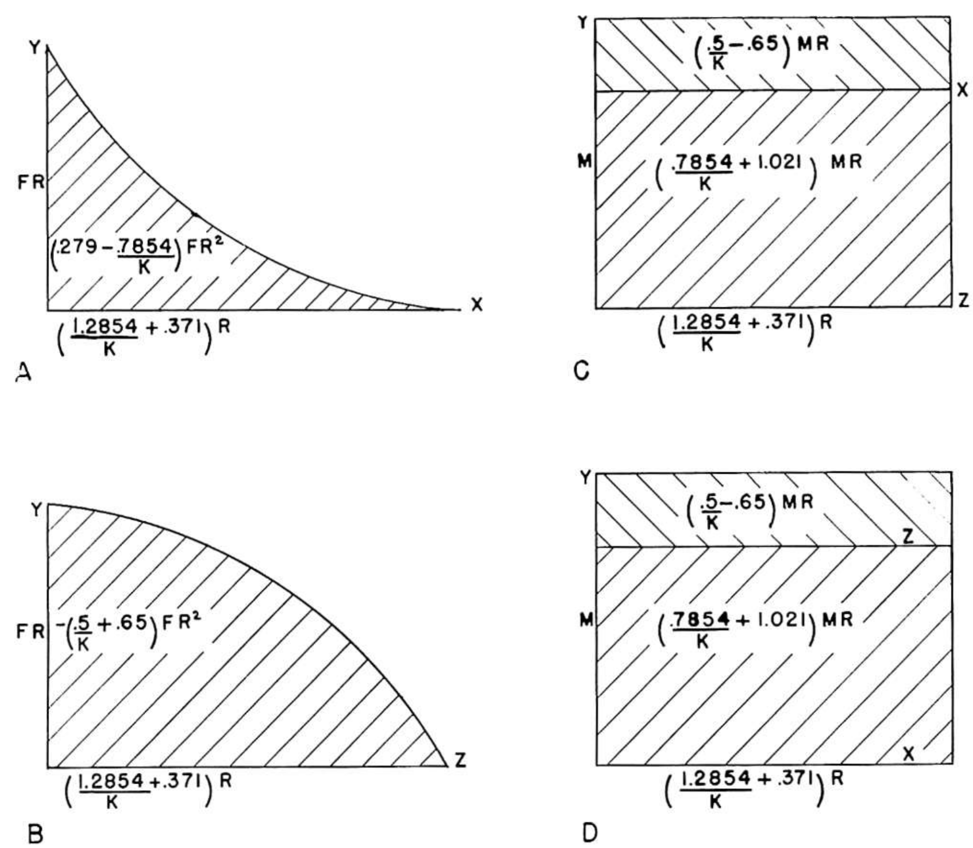

Fig. 23 Bending-Moment Diagrams for "Piping Handbook" Cases 8, 9, and 10
(Diagrams *A* and *B* apply to case 8; diagrams *C* and *D* apply to cases 9 and 10, respectively.)

contain a good percentage of their lengths in bends and that have large transverse moments along these bends are the only types in which large changes of end reactions need be expected.

It is hoped that this will give an idea of the quantitative influence of the transverse-rigidity factor as desired by Dr. Hovgaard. The stress concentration, caused by cross-sectional deformation, should be considered even though end-reaction changes are small. This is particularly true in view of the uncertain behavior of alloys used for high-temperature high-pressure piping at different stress combinations.

The application of the transverse-rigidity factor requires a negligible amount of additional work for most analytical methods of determining pipe end reactions. An appreciable added difficulty is encountered in applying it to the graphoanalytical method illustrated in the "Piping Handbook." In this latter case the abscissas of the bending-moment diagrams are taken, as stated by McCutchan, as the sum of the two rotations involved. For the handbook Cases 8, 9, and 10 the bending-moment diagrams are given in Fig. 23 of this closure, to supplement the illustrations given in the paper (Figs. 14, 15, and 16). If secondary rotations are to be neglected, the minor rotations in these diagrams can be omitted.

The additional unsolved problems put forth by Rossheim and Markl emphasize that considerable research must be accomplished before fundamental engineering knowledge catches up with present engineering practice as applied to piping. Rather than tending to discourage, a listing of these problems should provide greater incentive to labor. Many problems to be encountered along these lines are capable only of empirical solutions. Methods of measurements have lately been so improved that good results should be forthcoming. Perhaps a most important problem to be added to the list of Rossheim and Markl is the measurement of strains in installed piping systems which are caused by the temperature and pressure changes under service conditions. Such measurements are by no means impossible.

In Fig. 10 of the paper, a correction, in which S_L should be changed to S_t, should be noted. The curve for this figure is plotted for $h = 0.8$.