the last ten years.'' The original version of the paper did present considerable numerical examples and analyses, which, however, were dropped during the revision following reviewers' instructions that the paper was overlong. As a matter of fact, the published paper was still three pages over the limit, after considerable simplifications including the Introduction. As Dr. Velinsky points out, the relevant literatures are too numerous to mention. The author cannot see any reason to provide a complete publication list, in addition to those needed to be cited, especially in such an overlength situation. Dr. Velinsky then mentions a 1985 paper of his, and concludes that the ''nonlinear theory provides no value over the linear theory.'' The author disagrees with such a point of view. It is well known that almost all larger deformation problems encounter the possibility of plastic deformation. The elastic nonlinear theory, however, remains an important branch of mechanics. In general, Dr. Velinsky denies any contribution of the paper and only advocates his own accomplishment in this field. The author has no intention of making any comments on his claims, but believes that readers are professionals and have the best judgement.

Bifurcation of Orthotropic Solids[4]

A. Chattopadhyay[5,7] and H. Gu[6,7]. DeBotton and Schugasser (1996) recently presented an exact solution for bifurcation of orthotropic solids. In the paper, the following equilibrium equations were used.

$$\nabla \cdot [(\boldsymbol{\sigma} + \Sigma) \cdot \nabla(\mathbf{x} + \mathbf{u}) = \mathbf{0} \tag{1}$$

An assumption was made for plane-strain conditions that $\Sigma_{11} = -p$ is the only nonvanishing component of the initial stress in Eq. (1). Therefore, the remaining equilibrium equations can be derived by ignoring the product terms $\boldsymbol{\sigma} \cdot \nabla \mathbf{u}$ and their derivatives.

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} - p\,\frac{\partial^2 u_1}{\partial x_1^2} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} - p\,\frac{\partial^2 u_2}{\partial x_1^2} = 0 \tag{2}$$

However, it is our purpose to point out that their approach contains a fundamental error resulting from ignoring the primary prebuckling displacements, which normally provide the same order of contributions as prebuckling stresses at the buckling state. Of course, these prebuckling displacement contributions are not included in simplified plate-type theories for simplicity. For an exact elasticity solution, which is the motivation of DeBotton and Schugasser's work, these terms should be included to ensure a rigorous analysis procedure.

If the superscript $(\ )^0$ is used to denote the prebuckling terms, the primary prebuckling state of for an orthotropic half-space whose material axes are parallel to the geometric axes can be truly simulated by

$$\sigma_{11}^0 = -p, \quad \sigma_{22}^0 = \sigma_{12}^0 = 0. \tag{3}$$

Using the constitutive equation for this orthotropic half-space, the derivatives of prebuckling displacements are derived as follows:

$$\frac{\partial u_1^0}{\partial x_1} = -\frac{C_{22}}{C_{11}C_{22} - C_{12}^2}\,p, \quad \frac{\partial u_2^0}{\partial x_2} = \frac{C_{12}}{C_{11}C_{22} - C_{12}^2}\,p$$

$$\frac{\partial u_1^0}{\partial x_2} = \frac{\partial u_2^0}{\partial x_1} = 0. \tag{4}$$

Including these displacement contributions, the buckling equations can be finally stated as

$$\frac{\partial}{\partial x_1}\left[\sigma_{11}\left(1 + \frac{\partial u_1^0}{\partial x_1}\right)\right] + \frac{\partial}{\partial x_2}\left[\sigma_{12}\left(1 + \frac{\partial u_1^0}{\partial x_1}\right)\right] - p\,\frac{\partial^2 u_1}{\partial x_1^2} = 0$$

$$\frac{\partial}{\partial x_1}\left[\sigma_{12}\left(1 + \frac{\partial u_2^0}{\partial x_2}\right)\right] + \frac{\partial}{\partial x_2}\left[\sigma_{22}\left(1 + \frac{\partial u_2^0}{\partial x_2}\right)\right] - p\,\frac{\partial^2 u_2}{\partial x_1^2} = 0 \tag{5}$$

and the buckling equation can also be expressed in terms of displacements as follows:

$$\left(1 - \frac{C_{22}}{C_{11}C_{22} - C_{12}^2}\,p\right)\left[C_{11}\,\frac{\partial^2 u_1}{\partial x_1^2} + C_{66}\,\frac{\partial^2 u_1}{\partial x_2^2} + (C_{12} + C_{66})\,\frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right] - p\,\frac{\partial^2 u_1}{\partial x_1^2} = 0$$

$$\left(1 + \frac{C_{12}}{C_{11}C_{22} - C_{12}^2}\,p\right)\left[(C_{12} + C_{66})\,\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + C_{66}\,\frac{\partial^2 u_2}{\partial x_1^2} + C_{22}\,\frac{\partial^2 u_2}{\partial x_2^2}\right] - p\,\frac{\partial^2 u_2}{\partial x_1^2} = 0. \tag{6}$$

The prebuckling displacement contributions in these two equations can be written as

$$\Delta_1 = -\frac{C_{22}}{C_{11}C_{22} - C_{12}^2}\,p\left[C_{11}\,\frac{\partial^2 u_1}{\partial x_1^2} + C_{66}\,\frac{\partial^2 u_1}{\partial x_2^2} + (C_{12} + C_{66})\,\frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right]$$

$$\Delta_2 = \frac{C_{12}}{C_{11}C_{22} - C_{12}^2}\,p\left[(C_{12} + C_{66})\,\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + C_{66}\,\frac{\partial^2 u_2}{\partial x_1^2} + C_{22}\,\frac{\partial^2 u_2}{\partial x_2^2}\right]. \tag{7}$$

Therefore, their effects in the buckling analysis can now be discussed.

Orthotropic Material With $C_{11} \gg C_{22}$

In this extreme case, the values of the material properties are assumed to be

$$C_{11} \gg C_{22}, \quad C_{12} \cong \frac{C_{22}}{3}, \quad C_{66} \cong \frac{C_{22}}{2}.$$

The prebuckling displacement contributions in the buckling equations are thus simplified as follows:

$$\Delta_1 \cong -p\,\frac{\partial^2 u_1}{\partial x_1^2} - \frac{C_{22}}{2C_{11}}\,p\,\frac{\partial^2 u_1}{\partial x_2^2} - \frac{C_{22}}{C_{11}}\,p\,\frac{\partial^2 u_2}{\partial x_1 \partial x_2} \cong -p\,\frac{\partial^2 u_1}{\partial x_1^2}$$

$$\Delta_2 \cong \frac{C_{22}}{3C_{11}}\,p\,\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{C_{22}}{6C_{11}}\,p\,\frac{\partial^2 u_2}{\partial x_1^2} + \frac{C_{22}}{3C_{11}}\,p\,\frac{\partial^2 u_2}{\partial x_2^2} \cong 0. \tag{8}$$

[4] deBotton, G., and Schulgasser, K., 1996, ''Bifurcation of Orthotropic Solids,'' ASME JOURNAL OF APPLIED MECHANICS, Vol. 63, pp. 317–320.
[5] Associate Professor. Mem. ASME.
[6] Graduate Research Associate. Mem. ASME.
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Compared to the prebuckling stress contributions, $-p(\partial^2 u_1/\partial x_1^2)$ and $-p(\partial^2 u_2/\partial x_1^2)$, in Eq. (6), it is clear that the prebuckling displacements in the first equation of Eq. (6) provide at least similar magnitudes of contributions as the prebuckling stress in this case.

Isotropic Material

The values of the material properties in an isotropic case are assumed to be

$$C_{11} = C_{22}, \quad C_{12} \cong \frac{C_{22}}{3}, \quad C_{66} \cong \frac{C_{22}}{2}.$$

The prebuckling displacement effects in the buckling equations are now written as follows:

$$\Delta_1 \cong -p \frac{\partial^2 u_1}{\partial x_1^2} - \frac{1}{2} p \frac{\partial^2 u_1}{\partial x_2^2} - p \frac{\partial^2 u_2}{\partial x_1 \partial x_2}$$

$$\Delta_2 \cong \frac{1}{3} p \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{1}{6} p \frac{\partial^2 u_2}{\partial x_1^2} + \frac{1}{3} p \frac{\partial^2 u_2}{\partial x_2^2}. \qquad (9)$$

It can be concluded that, in this case, the prebuckling displacement contributions are at least the same as that due to the prebuckling stress in the first equation of Eq. (6) and are at least as much as one-sixth of that due to the prebuckling stress in the second equation of Eq. (6).

The contributions due to the prebuckling displacements, in cases other than the two cases discussed above, should be somewhere between these two extreme cases. Equations (6) were used by the authors (Chattopadhyay and Gu, 1996) in addressing the buckling of orthotropic plates and composite laminates. To our knowledge, excluding prebuckling displacement contributions will introduce as much as a ten percent error, depending upon the material properties and the length-to-thickness ratio of the plates.

Reference

Chattopadhyay, A., and Gu, H., 1996, "Exact Elasticity Solution for Buckling of Composite Laminates," *Composite Structures*, Vol. 34, pp. 291–299.

Authors' Closure[8]

A rigorous solution for the buckling of orthotropic solids was given in our paper. It was stated there explicitly that the equilibrium equation is expressed in terms of the second Piola-Kirchhoff stress which measures the tractions per unit area of the reference or undeformed configuration. Following Malvern (1969), in terms of the Piola-Kirchhoff stress $\tilde{\mathbf{T}}$, the equilibrium equation is

$$\nabla \cdot [\tilde{\mathbf{T}} \cdot \tilde{\mathbf{F}}^T] = \mathbf{0}, \qquad (1)$$

and it is clear that the deformation gradient $\mathbf{F}$ and hence the entire equation, will depend on the choice of the reference configuration. In some cases, such as the one considered by us, it is useful to let $\tilde{\mathbf{T}} = \boldsymbol{\sigma} + \boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma}$ is a uniform stress field and $\boldsymbol{\sigma}$ is a stress increment beyond this pre-existing stress, and also to substitute $\mathbf{F} = \nabla(\mathbf{x} + \mathbf{u})$, where $\mathbf{x}$ is the position vector and $\mathbf{u}$ is the displacement vector which is measured from the reference configuration. If we further assume that the stress increment is small, Eq. (1) may be rewritten in the form

$$(\nabla \cdot \boldsymbol{\sigma}) \cdot (\mathbf{1} + \nabla \mathbf{u}) + \boldsymbol{\Sigma} : \nabla\nabla \mathbf{u} = \mathbf{0}. \qquad (2)$$

In our work the reference configuration for the Piola-Kirchhoff stress was specifically chosen as the prebuckled state, or, quoting from the paper: "Initially, the solid is in a state of homogeneous strain corresponding to compressive uniform stress p parallel to the free boundary along the x_1 direction". We note that, since the displacements are measured from the prebuckled state, the components of the displacement gradient which is proportional to $\boldsymbol{\sigma}$ are small. Obviously, in this case, the product term $(\nabla \cdot \boldsymbol{\sigma}) \cdot \nabla \mathbf{u}$ is small in comparison with other terms in the equation and may be neglected. If, on the other hand, another reference configuration is chosen and the displacements are measured from it, then, the term $(\nabla \cdot \boldsymbol{\sigma}) \cdot \nabla \mathbf{u}$ must be taken into account. As a result, terms that depend on the history of the deformation will appear explicitly in the equilibrium equation. Nonetheless, it should be realized that these terms appear solely due to the fact that Eqs. (1) and (2) are expressed in terms of the pseudo Piola-Kirchhoff stress and not in terms of the true Cauchy stress.

We note that, from a purely rigorous mathematical point of view, the choice of the reference configuration is immaterial. However, we also note that there exist good motivations in favor of choosing the reference configuration as the prebuckled configuration. First, this choice results in strictly simpler expressions because the term $(\nabla \cdot \boldsymbol{\sigma}) \cdot \nabla \mathbf{u}$ is neglected. Further, and even more important, is the fact that by adopting this approach the problem can be solved without the need to account for the entire history of the deformation up to the instant when the material buckles. Thus, as was used in our paper, the instantaneous stress-strain relation at the prebuckled configuration is all that is needed to obtain a rigorous solution for the problem, or in the words of our paper, "we assume a linear relationship between the stress increment and the *infinitesimal* strain increment". This feature enables utilizing the proposed solution to predict the buckling stress of nonlinearly deforming materials. In fact, while the instantaneous stress-strain relation at any point of the deformation history may be linearized, it is well known that when the deformation gradient is large, the overall or total stress-strain relationship is usually nonlinear. This, of course, provides an additional motivation for choosing the reference configuration as the prebuckled configuration. At the same time it renders the approach of Chattopadhyay and Gu tenuous at best.

It is possible that in some special cases a different reference configuration will be a more appropriate one. However, since in such cases the deformation history will be incorporated into the solution, it will be valid only to the particular class of materials that was considered. But, if such an approach is taken, it will generally not be adequate to utilize a linear stress-strain relationship and thus the analysis of Chattopadhyay and Gu is an unwarranted simplification. Finally, we note that in some cases, usually when the longitudinal Young's modulus is much larger than the other moduli, the critical buckling strain is small and the assumption of a linear stress-strain relationship is satisfactory. In these cases, since the strains are small, the choice of the reference configuration is unimportant and hence, for the sake of simplicity alone, the formulation that was proposed by us is an advantageous one.

Reference

Malvern, L. E., 1969, *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Englewood Cliffs, NJ.

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