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Continuation of Newton's Method Through Bifurcation Points[1]

R. SCHMIDT[2] and D. A. DaDEPPO.[3] It is well known that the Chebyshev polynomial approximation to a given function in a prescribed interval tends to minimize the maximum absolute error of the approximation in this interval [1].[4] It has also been shown [2, 3] that Chebyshev's polynomial approximations of the nonlinear restoring functions lead to very accurate approximate values for the period of oscillation of a single-degree-of-freedom system. Hence, instead of expanding $\sin \theta$ in a Taylor series about θ_1 in equation (4), we may find it remunerative to represent $\sin (\theta_1 + y)$ by the series of Chebyshev polynomials in the interval $-\alpha \leqslant y \leqslant \alpha$. Thus [4], through the linear terms,

$$\sin (\theta_1 + y) = \sin \theta_1 \cos y + \cos \theta_1 \sin y$$

$$\approx J_0(\alpha) \sin \theta_1 + \frac{2}{\alpha} J_1(\alpha) y \cos \theta_1 \quad (1)$$

The choice of interval $[-\alpha, \alpha]$ is justified by the identical behaviors of the cantilever and the two-hinged column of length $2L$.

For simplicity, let us assume $\theta_1 = \theta_0 = 0$, so that $y = \theta$ and, in view of equation (1), the governing equation is written

$$\theta'' + k^2\theta = 0 \quad (2)$$

where

$$k^2 = \frac{2\lambda}{\alpha} J_1(\alpha), \qquad \alpha = \theta(0) \quad (3)$$

and

$$\theta = \alpha \cos kz \quad (4)$$

satisfies equation (2) and the boundary conditions, provided that $k = 1$; i.e.,

$$\lambda \equiv \frac{P}{P_{cr}} = \frac{\alpha}{2J_1(\alpha)} \quad (5)$$

Equation (5) is an approximate nonlinear relation between the applied load P and the rotation $\alpha \equiv \theta(0)$ of the free end. For example, if $\lambda = 1.293$, $\alpha = 1.403$ as compared to the author's value of $\alpha = 1.340$ and the exact value of $\alpha = 1.396$. More extensive numerical results are presented in [5].

The foregoing presentation does not detract anything from the author's method, which is of a more general nature.

[1] By G. A. Thurston, published in the September, 1969, issue of the Journal of Applied Mechanics, Vol. 36, Trans. ASME, Vol. 91, Series E, pp. 425–430.

[2] Professor of Engineering Mechanics, University of Detroit, Detroit, Mich. Mem. ASME.

[3] Professor of Civil Engineering, University of Arizona, Tucson, Ariz.

[4] Numbers in brackets designate References at end of Discussion.

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Author's Closure[5]

The purpose of the paper was to outline the procedure to follow in applying Newton's method in the neighborhood of bifurcation points and limit points. In the example of the imperfect column problem, only terms up to the third power in A were retained in the numerical computations involving $\cos (A \cos z)$ and $\sin (A \cos z)$. This was done with the aim of making the discussion of the method easier to follow. However, this truncation led Professor Schmidt and Professor DaDeppo to the erroneous conclusion that the Taylor series expansion that is used to start Newton's method in equation (4) of the paper[6] is inaccurate.

Newton's method is based on the assumption that a sequence of linear differential equations can be solved. In practice, most linear problems cannot be solved in closed form. This does not make Newton's method inaccurate. A review of the analysis for the perfect column will illustrate this point.

To solve the differential equation

$$\theta'' + \lambda \sin \theta = 0$$

with boundary conditions

$$\theta'(0) = \theta(\pi/2) = 0$$

by Newton's method, assume a solution θ_1 and seek a correction y such that

$$\theta = \theta_1 + y$$

Substituting in the differential equation and expanding $\sin \theta$ in a Taylor's series about θ_1 leads to equation (4) in the paper. Away from bifurcation points, it is sufficient to retain only linear terms in y so that the equation to be solved for y is

$$y'' + \lambda(\cos \theta_1)y = -(\theta_1'' + \lambda \sin \theta_1)$$

When

$$\theta_1 = A \cos z$$

the equation becomes

$$y'' + \lambda[J_0(A) - 2J_2(A) \cos 2z + 2J_4(A) \cos 4z - \ldots]y$$

$$= [A - 2\lambda J_1(A)] \cos z + 2\lambda[J_3(A) \cos 3z - J_5 \cos 3z + \ldots]$$

In the paper, only terms up to the third power in A were retained in the previous equation, leading to a 4 percent error when A is 80 deg.

The right side of the equation is the error in the assumed solu-

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[6] There is a typographical error in equation (4). The coefficient of y^3 should have a positive sign. Also, the first term in equation (21*a*) should be $L(F)$.

tion θ_1. The constant A can be selected to make the first term vanish or nearly vanish

$$\frac{A}{2J_1(A)} = \lambda$$

The foregoing equation corresponds to equation (5) of the discussers.

The particular solution for y, which also satisfies the boundary conditions, can be computed by Galerkin's method from the series

$$y = \sum_{n=1} C_{2n-1} \cos (2n - 1)z$$

If only two terms are retained in the series, we have

$$[J_0(A) - J_2(A) - 1/\lambda]C_1 - [J_2(A) - J_4(A)]C_3 = [A/\lambda - 2J_1(A)]$$

$$-[J_2(A) - J_4(A)]C_1 + [J_0(A) - J_6(A) - 9\lambda]C_3 = 2J_3(A)$$

when $A = 1.40$ and $\lambda = 1.29389$, the result is

$$\theta = 1.40 \cos z + 0.01233 \cos z - 0.01619 \cos 3z$$

$$\theta(0) = 1.39614$$

as compared to the exact result

$$\theta(0) = 1.39626$$

More accuracy for y can be obtained by taking more terms in the series solution. To improve the solution for θ, set

$$\theta_2 = (\theta_1 + y)$$

and repeat the foregoing procedure to obtain y_2, the correction to θ_2,

$$y_2'' + \lambda(\cos \theta_2)y_2 = -(\theta_2'' + \lambda \sin \theta_2) = \frac{\lambda(\sin \theta_1)y^2}{2} - \dots$$

A general conclusion can be drawn for the remarks of Professor Schmidt and Professor DaDeppo and this Closure. In starting the Newton's method iteration, it is necessary to have a first guess at the solution. One possible choice is a function containing a free parameter (A in the example). The free parameter can then be set by minimizing the error that appears on the right side of the linearized equations over the interval of integration. This minimization can be only approximate, as it was in the foregoing example.