Table 1 Parametric resonance and combination resonance of sum and difference type (ω_i: eigenfrequency)

Mode no.	1	2	3	4	5
1	$2\omega_1$	$\omega_2 - \omega_1$	$\omega_1 + \omega_3$	$\omega_4 - \omega_1$	$\omega_1 + \omega_5$
2		$2\omega_2$	$\omega_3 - \omega_2$	$\omega_2 + \omega_4$	$\omega_5 - \omega_2$
3			$2\omega_3$	$\omega_4 - \omega_3$	$\omega_3 + \omega_5$
4				$2\omega_4$	$\omega_5 - \omega_4$
5		symmetry			$2\omega_5$

Second, it is concluded that for the cantilevered pipes the parametric resonances are selectively associated with only some of the modes of the system. Is it the general conclusion for a pipe conveying pulsating fluid? If it is, I would like to know the reason.

Authors' Closure

We are very grateful to Professor Iwatsubo for his discussion of our paper.

Concerning his first point, there appears to be some contradiction between the second and third paragraphs of the Discussion. However, we agree with the first statement made by the discusser that, for cantilevered columns, both sum and difference-type combination resonances are possible. In our paper, concerning cantilevered pipes conveying fluid, we were careful to say that "the combination resonances *appear* to involve the differences, rather than the sums." We have not made a special study of this, and our supposition was based on the consideration that for the low frequencies involved it is more likely that the combination resonances be of the difference rather than the sum type, since in the latter case $(\omega_i + \omega_j)/k$ near zero would imply very large values of k.

Concerning the second point, the limited extent of our calculations does not allow us to say with certainty that, let us say, first-mode parametric resonances are impossible for *all* possible sets of system parameters. However, we have never found such instabilities in our analysis, nor has it ever been found in the experiments [1].[3] A possible explanation is this. In the cantilevered pipe the Coriolis acceleration acts effectively as a damping force and the effective damping varies from one mode to another [2]. It is certainly possible that some modes are simply too heavily "damped" by the Coriolis effect to exhibit parametric resonances, either over a wide range of flow velocities or for all flow velocities.

References

1 Paidoussis, M. P., and Issid, N. T. "Experiments on Parametric Resonance of Pipes Containing Pulsatile Flow," JOURNAL OF APPLIED MECHANICS Vol. 43, No. 2, TRANS. ASME, Vol. 98, Series E, pp. 198–202.

2 Paidoussis, M. P., and Issid, N. T. "Dynamic Stability of Pipes Conveying Fluid," *Journal of Sound and Vibration*, Vol. 33, 1974, pp. 267–294.

[3] Numbers in brackets designate References at end of Closure.

Measurement of Angular Acceleration of a Rigid Body Using Linear Accelerometers[1]

Y. King Liu.[2] The authors are to be commended for posing a most interesting inverse problem in rigid body mechanics, i.e., is it possible to infer the total acceleration of a rigid body from strategically placed linear accelerometers? The writer wishes to indicate however that the equations (3)–(5) of the authors' paper in fact can be shown analytically to be unstable. Equations (3)–(5) can be rewritten as

$$\dot{\omega}_x = \alpha_x - \omega_y\omega_z \qquad (a)$$

$$\dot{\omega}_y = -\alpha_y + \omega_x\omega_z \qquad (b)$$

$$\dot{\omega}_z = \alpha_z - \omega_x\omega_y \qquad (c)$$

where $\alpha_x = (A_{z1} - A_{z0})/\rho_{y1}$, $\alpha_y = (A_{z2} - A_{z0})/\rho_{x2}$ and $\alpha_z = (A_{y2} - A_{y0})/\rho_{x2}$. The stability analysis of the system of equations just given can be obtained through the calssical Routh-Hurwitz criterion [1].[3] A state of equilibrium may be represented by a singular point, i.e., where all the derivatives of the dependent variable with respect to time are simultaneously zero, i.e.,

$$\alpha_x - \omega_y^0\omega_z^0 = 0 \qquad (d)$$

$$-\alpha_y - \omega_x^0\omega_z^0 = 0 \qquad (e)$$

$$\alpha_z - \omega_x^0\omega_y^0 = 0 \qquad (f)$$

where ω_x^0 ω_y^0 and ω_z^0 denote a set of equilibrium values for the dependent variables. Consider small perturbations, ξ_i, defined by the following equation:

$$\omega_x = \omega_x^0 + \xi_x \qquad (g)$$

$$\omega_y = \omega_y^0 + \xi_y \qquad (h)$$

$$\omega_z = \omega_z^0 + \xi_z \qquad (i)$$

Substituting the foregoing into (a)–(c) yields the following:

$$\dot{\xi}_x = \omega_y^0\xi_x - \omega_z^0\xi_y \qquad (j)$$

$$\dot{\xi}_y = \omega_x^0\xi_z + \omega_z^0\xi_x \qquad (k)$$

$$\dot{\xi}_z = -\omega_x^0\xi_y - \omega_y^0\xi_x \qquad (l)$$

It has been shown by Liapunov [2] that if the real parts of the roots of the characteristic equation of the foregoing system are negative then the corresponding equilibrium state is stable or if at least one root has a positive real part the equilibrium is unstable. The characteristic equations may now be written as follows:

$$\begin{vmatrix} -\lambda & -\omega_z^0 & -\omega_y^0 \\ \omega_z^0 & -\lambda & \omega_x^0 \\ -\omega_y^0 & -\omega_x^0 & -\lambda \end{vmatrix} = 0$$

or

$$-\lambda^3 + (-\omega_x^{02} + \omega_y^{02} + \omega_z^{02})\lambda + 2\omega_x^0\omega_y^0\omega_z^0 = 0 \qquad (m)$$

The Routh-Hurwitz criterion requires that all coefficients in the characteristic equation exist and are positive. In (m), the λ^2 term is missing. Thus the given set of differential equations is *unstable*.

Since the solution to the foregoing set of equations is unique given the initial values, in spite of its being unstable, one can further assert that no amount of additional manipulation or clever placement of the six accelerometers would yield a set of equations which will be stable.

By going to the nine accelerometer scheme the authors have in fact added 3 pieces of information, i.e., equations (6)–(8) to the original (3)–(5). The authors' equations (9)–(11) now contain 9 temporal measurements and straightforward algebra to obtain the desired result. The major disadvantage is obviously the addition of 3 channels of data acquisition and analysis.

[1] By A. J. Padgaonkar, K. W. Krieger, and A. I. King, and published in the September, 1975, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 42, TRANS. ASME, Vol. 97, Series E, pp. 552–556.

[2] Professor and Director, Biomechanics Laboratory, Tulane University Medical School, New Orleans, La.

[3] Numbers in brackets designate References at end of Discussion.

In spite of the fact that the system of equations (3)–(5) is unstable, it might still be possible to utilize these results in a short duration impact because instabilities take time to grow. It is conceivable that a scheme could be devised in which the growth rate of the dependent variables are as slow as possible, i.e., make the positive real part of the characteristic root as small as possible so that its effects in a short duration impact are negligible.

References

1 Routh, E. J., *Dynamics of Rigid Bodies*, Dover Publications, Inc., New York, 1955; reprint of 1892 edition of MacMillan & Co., Ltd., London.

2 Liapunov, M. A., "Problème général de la stabilité du mouvemant" *Ann. Fac. Sci. Univ. Toulouse*, Vol. 9, 1907, pp. 203–469; *Ann. Math. Studies*, No. 17, Princeton University Press, Princeton, N.J., 1961.

Authors' Closure

The authors are indeed pleased that Dr. Liu has provided analytical support to the experimental findings discussed in the paper. While it is important to show that the use of six linear accelerometers can lead to erroneous values of angular acceleration, it should be pointed out that the stability analysis provided by Dr. Liu considers a special configuration of six accelerometers and that it may not be valid for the general case of arbitrarily located accelerometers. In the latter situation, the governing equations become quite complex and a linearized perturbation analysis cannot definitely establish the instability of the system. In fact, the results in the discussion show that the solution is not asymptotically stable and are not sufficient to prove that it is unstable.

With regard to the growth of instability it should be noted that the problem at hand is the accurate determination of angular acceleration. If the system of equations is indeed unstable, then error is present as soon as $t > 0$. Furthermore, it is only in very special cases that one can insure a slow growth rate, since the characteristic roots are dependent upon the magnitude and sign (direction) of the angular velocity components of a rigid body in general three-dimensional motion.

Finally, the burden of proof of stability rests with the six-accelerometer user who must also identify quantitatively the time beyond which the errors become intolerably large.

Scattering of Water Waves by a Pair of Semi-Infinite Barriers[1]

G. Dagan.[2] The independence of the transmission coefficient T upon the angle of incidence α of the far wave, which represents one of the main results obtained by the author, seems to be in contradiction with simple physical facts.

Indeed, let us consider an incident wave with crest normal to the breakwaters, i.e., with $\alpha = 0$. In this case there is no scattering and $\phi_s = 0$ (eq (7)) while $\phi_i = A \exp(-ikx)$. It is obvious that an exact solution for the wave propagated along the channel is $\phi = \phi_i$ and the transmission coefficient is exactly $T = 1$. Hence, the solution for (T) (equation (27)) cannot hold in this case and it is doubtful that it applies to other angles of incidence as well.

Author's Closure

G. Dagan raised an important point on the uniform validity of the asymptotic solution; in particular, for the normal incidence, i.e., $\alpha \to 0$. The author agrees with him that in this case the transmission coefficient should be exactly $T = 1$. Due to the fact that the scattered waves (equation (8)) were in the order of magnitude of $O(\sqrt{kr})$ and were neglected in the ϕ analysis, the transmission coefficient (equation (26)) could be best interpreted as $T = 1 + O(\sqrt{ka})$. In other words, as $\sqrt{ka} \ll 1$ the transmitted waves between two breakwaters are indeed independent of the angle of incidence.

To include the effects of the angle of incidence, the scattered waves should be included and modifications on the inner solutions are needed. The author has not so far completed this study and would consider the question an open one.

[1] By P.L.-F. Liu and published in the December, 1975, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 42, No. 4, TRANS. ASME, Vol. 97, Series E, pp. 777–779.

[2] Professor of Fluid Mechanics, Technion, Israel Institute of Technology, Haifa, Israel.

A Practical Two-Surface Plasticity Theory[1]

A. Phillips.[2] The author should be congratulated for a very interesting and stimulating paper. Three comments should be added to the author's presentation. The two most important ones are first that according to a large number of experimental results, by the reviewer and his coworkers, some of which have already been published [1–3][3] and some of which are still in the process of publication [4], the theory of Mroz, at least as used in the general formulation of the present paper, does not agree with the experimental results. Not only the form of the yield surface changes with the motion but also the center of the yield surface does not move in the direction indicated by the author. The law of hardening of the yield surface is still not clear. The limit surface on the other hand could be considered to grow isotropically from the initial yield surface with its center remaining unchanged.

The second comment is that the stress-strain curves, the modeling of which is attempted by the author, include rate effects and therefore cannot be modeled very well by a plasticity theory. A theory of viscoplasticity is more likely to be successful. In particular, a theory of plasticity will represent the gross behavior of the stress-strain curves, while a theory of viscoplasticity will be able to represent the exact form of the curves.

The third comment is that the concept of the two surfaces plasticity theory has some previous history. It was considered first by this reviewer [5] and elaborated in a number of subsequent publications by him and his coworkers [6–8].

All three comments do not distract seriously from the achievement of the author in presenting a complete theory which, however, requires extensive improvements to become useful for the practice.

References

1 Phillips, A., and Tang, J. L., "The Effect of Loading Path on the Yield Surface at Elevated Temperatures," *International Journal of Solids and Structures*, Vol. 8, 1972, pp. 463–474.

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6 Eisenberg, M. A., and Phillips, A., "A Theory of Plasticity With Non-coincident Yield and Loading Surfaces," *Acta Mechanica*, Vol. 11, 1971, pp. 247–280.

[1] By R. D. Krieg, and published in the September, 1975, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 42, No. 3, TRANS. ASME, Vol. 97, Series E, pp. 641–646.

[2] Professor of Engineering and Applied Science, Yale University, New Haven, Conn. Fellow ASME

[3] Numbers in brackets designate References at end of Discussion.