frequencies than the Rayleigh-Ritz method of the authors' reference [1]. One minor advantage of the finite-element method is that it can be applied to shells with any end conditions without adding to the complexity of the computation; whereas with the authors' method a change from simply supported ends would require the use of more complicated functions for $\psi_{mn}(x)$, $\eta_{mn}(x)$, and $\xi_{mn}(x)$.

Authors' Closure

The authors would like to extend their appreciation to Professor Egle and Professor Warburton for their interesting comments.

With respect to Professor Egle's first comment, we would like to point out that the equations resulting from the authors' formulation, equations (22), are not the same as those obtained in the discusser's reference [2]. For, the authors have represented the deflections of the stiffened shell in terms of the actual modes of free vibration of the unstiffened shell, while in the discusser's analysis, the corresponding representation of these deflections are given in terms of a Fourier series in u, v, w. True, the functions employed are those used to describe the modes of the unstiffened shell; however, in the discusser's work, the coefficients of superposition in these series are independent parameters. The distinction between the two representations is made clear by rewriting them. In the authors' paper, in the case of the freely supported circular cylindrical shell, equation (10) for the deflections of the stiffened shell, becomes

$$
\begin{aligned}
u_s &= \sum_{k=1}^{3} \sum_{m} q_{mn}{}^k(t)[(U_{mn}{}^k/W_{mn}{}^k) \cos (m\pi x/l)] \cos n\theta \\
v_s &= \sum_{k=1}^{3} \sum_{m} q_{mn}{}^k(t)[(V_{mn}{}^k/W_{mn}{}^k) \sin (m\pi x/l)] \sin n\theta \qquad (1) \\
w_s &= \sum_{k=1}^{3} \sum_{m} q_{mn}{}^k(t)[\sin (m\pi x/l)] \cos n\theta
\end{aligned}
$$

where the $q_{mn}{}^k(t)$ are generalized coordinates, and the ratios $(U_{mn}{}^k/W_{mn}{}^k)$ and $(V_{mn}{}^k/W_{mn}{}^k)$ are the amplitude ratios determined by the $k = 1, 2,$ or 3 natural mode of the unstiffened shell, vibrating in a configuration consisting of m half waves in the axial, and n full waves in the circumferential direction. In Professor Egle's reference [2], the displacements of the stiffened shell are taken as

$$
\begin{aligned}
u &= \sum_{m} \sum_{n} [\bar{u}_{mn} \cos n\theta + \bar{u}_{mn}{}' \sin n\theta] \cos (m\pi x/l) \sin \omega t \\
v &= \sum_{m} \sum_{n} [\bar{v}_{mn} \sin n\theta - \bar{v}_{mn}{}' \cos n\theta] \sin (m\pi x/l) \sin \omega t \qquad (2) \\
w &= \sum_{m} \sum_{n} [\bar{w}_{mn} \cos n\theta + \bar{w}_{mn}{}' \sin n\theta] \sin (m\pi x/l) \sin \omega t
\end{aligned}
$$

where the terms involving primed and unprimed coefficients differentiate modes symmetric and antisymmetric with respect to a given reference plane. For free vibrations of axisymmetric shells, attention may be confined to unprimed terms, and by considering one circumferential mode at a time, the summation on n ignored. Now, the consequence of the difference between the author's and the discusser's representation is this: In the authors' analysis, the addition of a term in the series representation of the deflections involves the addition of a single parameter corresponding to a mode of the unstiffened shell, in which there is a definite ratio of the amplitudes of u, v, w, see equations (1) previously mentioned. In the analysis corresponding to the foregoing equations (2), the addition of a term involves the addition of three independent parameters corresponding to three different Fourier series. Thus the author's formulation and the discusser's formulation do not lead to the same set of equations.

Some further remarks which might be made here concern the equivalence of the extremum and equilibrium problems. If the set of admissible functions which are used to yield an extremum (for the potential energy in this case) also satisfy the equivalent system of differential equations and physical as well as geometric boundary conditions, then the formulation of the extremum and equilibrium problems become identical. Under such circumstances a formulation resulting from a Rayleigh-Ritz procedure can be identical with a formulation based on the authors' technique.

Professor Egle's comment concerning convergence is significant. His results bring striking attention to a possible pitfall. Results for the cases reported in the authors' paper were obtained by carrying the computations out to terms involving 30 modes of the unstiffened shell. In these cases convergence was usually approached after several terms. Upon becoming aware of the discusser's comment, the authors performed computations for a shell having the same geometric and physical properties as the discusser's, stiffened by 13 equally spaced rings. These rings, although of rectangular cross section, possessed moments of inertia identical to those chosen by the discusser. The results of the authors' computations essentially confirm those reported by Professor Egle.

The computations for the freely supported circular cylindrical shell were made principally to illustrate the application of the proposed technique, and consequently were limited in scope. The authors are thankful to Professor Warburton for employing those results and drawing on his own work to shed additional light on several aspects of this problem.

In conclusion, we would like to recall that at the heart of the technique proposed by the authors is the analysis of the unstiffened shell. That is, the natural modes of the unstiffened shell are used to construct the solution to the vibration problem of the stiffened shell. This progression from the solution of the unstiffened configuration, in addition to being an adjunct to the analysis, is part of the natural order of the investigation of a problem into which an additional complexity has been introduced (the stiffening elements). For, in connection with vibrational characteristics, the choice of stiffening elements depends strongly upon a knowledge of the behavior of the unstiffened structure. Solutions for a number of classes of unstiffened shells are already available in the literature.

The Linearization of the Prebuckling State and Its Effect on the Determined Instability Loads[1]

W. T. KOITER.[2] The main purpose of the paper is clearly reflected by its title, and the results obtained show that great caution should be exercised in linearizing the prebuckling state in problems where the nonlinear character of the behavior prior to buckling may be significant. This conclusion is, of course, hardly unexpected, but the detailed numerical results in the present paper are valuable because they indicate that the errors implied by the linearization of the prebuckling state may be quite substantial. In the two problems examined by the authors, the calculation of critical loads on the basis of a linearized prebuckling state results in an (unsafe) overestimate of the critical load.

In view of the scarcity of (more or less) "exact" analyses of postbuckling behavior of a continuous system in the literature mentioned by the authors, it may be worthwhile to draw atten-

[1] By A. D. Kerr and M. T. Soifer, published in the December, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 775–783.

[2] Professor, University of Technology, Delft, The Netherlands.

tion to some additional references [1, 2].[3] The papers by Biezeno are particularly relevant in this connection.

The primary purpose of the present discussion, however, is to explain the discusser's disagreement with a subsidiary conclusion of the authors, expressed in the last two paragraphs of the section "Conclusions," as well as in the body of the paper and the last two sentences of the summary. The point made by the authors may be formulated in the following way. In the bifurcation occurring in their problems a *linear* relationship (to a first approximation) exists between the magnitude of the skew-symmetric buckling mode and the change in external load. The authors conclude from this undisputed fact that "the usual adjacent equilibrium argument presented in the literature according to which only the displacements are perturbed, is not applicable for the determination of the bifurcation pressures of the shallow arch" (quotation from summary). This conclusion, however, seems to be incorrect. In fact, from the authors' own analysis of the bifurcation pressure, leading from their equation (77) to equation (94) it appears that condition (94) for the bifurcation point is obtained by putting the change in pressure $\hat{p}_0$ equal to zero.

In other words, the correct bifurcation pressure is indeed obtained, even if the change in load following the postbuckling range is ignored. If one is only interested in the bifurcation point, and not in the postbuckling behavior, the usual "adjacent equilibrium" argument is actually applicable (provided the system is, of course, conservative).

The mathematical reason for the validity of the usual adjacent equilibrium procedure in the determination of a bifurcation point is the nonuniqueness of displacement *increments* for a well-defined (infinitesimal) load increment at the bifurcation point. This lack of uniqueness implies that the *linear operator* L in the equation for the incremental displacement field $\hat{u}$

$$L\hat{u} = \hat{p},$$

where $\hat{p}$ represents the load increment, is a *singular* operator with nonvanishing solutions of the homogeneous equation

$$L\hat{u} = 0.$$

The latter equation is precisely the mathematical expression of the adjacent equilibrium argument.

References

1 Biezeno, C. B., "Ueber eine Stabilitätsfrage beim gelenkig gelagerten, schwachgekrümmten Stabe," *Proceedings of the Royal Netherlands Academy of Science*, Vol. 32, 1929, p. 990; "Das Durchschlagen eines schwach gekrümmten Stabes," *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 18, 1938, p. 21. Cf. also Biezeno, C. B., and Grammel, R., *Technische Dynamik*, Springer-Verlag, Berlin, Vol. 1, 1939, pp. 526–538.

2 Timoshenko, S. P., and Gere, J. M., *Theory of Elastic Stability*, 2nd ed., McGraw-Hill, New York, 1961, pp. 305–310.

E. F. MASUR.[4] The customary approach to the problem of determining the stability of an equilibrium configuration involves the superposition of a small disturbance on that configuration. The ensuing system of equations is then linear in terms of the disturbed displacements; however, the coefficients, which are often complicated functions of the space variables, require the prior solution of the basic nonlinear equations governing the configuration itself. In view of the mathematical complexity of the problem it is therefore reasonable to search for suitable simplifications. In particular, it appears reasonable to scrutinize the effect of the nonlinearity of the basic equations.

This nonlinearity generally arises from two possible sources. It may be of a constitutive nature, especially if "soft" materials are involved. However, in the present paper, as in most papers of this type, the assumption is made that the strain energy is quadratic in the (properly defined) strain components, and that therefore the stress components (again, after proper definition) are linearly related to the strain components.

The other source of nonlinearity, which applies also to the present paper, is kinematic in origin. If the strain-displacement relations contain quadratic terms, then a variational process applied to the potential energy leads to the equations of equilibrium as referred to the actual configuration. If the quadratic terms are deleted the same procedure yields the equations of equilibrium in the unstrained configuration. In other words, within the context of the current paper, the question of including or deleting nonlinear terms in the strain-displacement equations is tantamount to a discussion of the effect of the prebuckling deformation on the buckling characteristics of the structure.

Normally, this effect is negligible. For example, the buckling of a column is governed by the well-known Euler formula, which ignores the shortening of the column prior to buckling. Similarly, rings under external pressure are analyzed as if the radii, at the instant of buckling, were the original radii. Even arches or shells may often be analyzed under this assumption, although the actual boundary conditions may complicate the state of stress in the structure at the instant of buckling.

If prebuckling deformations are to be significant then the structure should exhibit a certain measure of geometric pathology; that is, an arch should be "shallow." It then becomes even shallower before it buckles, and, as expected and as demonstrated again in the present paper, to neglect this effect is to place oneself on the side of unrealistic optimism. Shallow shell caps fall in the same category, and the linkage, shown in the authors' Fig. 1, might again be considered shallow if $\theta_0 = 15$ deg, as indicated.

The question raised in the present paper has been the subject of two recent studies. Thompson, reference [3][5] of the Discussion, has made a thorough investigation of the effect of nonlinearities on the buckling characteristics of structures with finite degrees of freedom. The first part of the current paper should therefore represent a special case of that previous analysis, and it would be interesting to note how the authors' results compare with the broader conclusions established in the discusser's reference [3].

Another recent investigation of the effect of prebuckling deformations was carried out by Masur and Schreyer, reference [4] of the Discussion. This study, which was conducted within the framework of general continuum mechanics, went well beyond the objective of the present paper. Since the linearized version of the analysis may lead to unacceptable error and the exact version may run into unsurmountable mathematical difficulties, a perturbation technique has been suggested which may greatly increase the accuracy of the linear analysis while bypassing the need to solve a nonlinear set of basic equations.

What is of further interest herein is that the shallow arch employed as a demonstrative example by the present authors was also used in reference [4] of this Discussion, and for the same purpose. Moreover, the exact solution of the problem had already been obtained previously by Schreyer and Masur, the authors' reference [22].[6] It is therefore not surprising that the results given in Fig. 5 of reference [4] of this Discussion (allowing for a slight change in the definition of the variables) are in fact identical with Fig. 6 of the present paper, with respect to both

[3] Numbers in brackets designate References at end of Discussion.

[4] Head, Department of Materials Engineering, University of Illinois at Chicago Circle, Chicago, Ill.

[5] Numbers in brackets designate Additional References at end of Discussion.

[6] Except for modifications in the notation the exact analysis and results of present paper appear to be identical with [22]. However, it has been shown in [22] that while unsymmetric buckling is indeed possible for $K > 5.02$, (as claimed in the present paper) the point of bifurcation (point A) may lie on the descending branch of the curve. In that case, and unless initial imperfections change the picture, the arch snaps through symmetrically when the value of the load reaches P_u even though P_u is greater than the nonsymmetric buckling load P_{cr}. The latter governs when point A lies on the ascending branch of the curve; this happens for $K > 5.74$.

exact and approximate analyses. In addition, Fig. 5 of the previous paper also shows the improvement brought about through the introduction of one more term in the perturbation expansion developed in that paper.

On several occasions the present authors emphasize the point that the results of their analysis contradict the "adjacent equilibrium argument encountered usually in the literature." Since this argument has been developed on a very broad basis, reference [5–7] of this Discussion, and appears to cover all problems of elastic bodies under conservative loading conditions, it would indeed be surprising if the shallow arch were to be excluded from its range of applicability. The remainder of this Discussion is intended to show that this is in fact not the case and that the apparent contradiction seems to be based on a misinterpretation of the argument itself.

The question of buckling, or instability, is inherently a dynamic one and has no *a priori* connection with "neighboring configurations" or "points of bifurcation." If the problem is conservative, however, it is readily shown[7] that stability, that is (loosely), small responses to small disturbances, is assured provided the potential energy V of a configuration in equilibrium with a given load is less than that of any neighboring configuration (which need not be in equilibrium with that load). It is for this reason that static methods of the type discussed by the authors have been widely in use.

Specifically, if, following Koiter [[8] of this Discussion], the potential energy associated with any kinematically admissible configuration is compared with the potential energy of an equilibrium configuration (State I), then the difference ΔV may be expressed in the form

$$\Delta V = V_2 + V_3 + V_4 + \ldots, \tag{1}$$

in which V_2 is quadratic in the relative displacements, V_3 cubic, etc. The two configurations and the corresponding potential energy levels are associated with the same load parameter. As expected, the linear term in the discusser's equation (1) is missing because State I is in equilibrium.

This equilibrium is then locally stable if $V_2 > 0$ for all possible (not identically vanishing) displacements, and locally unstable if $V_2 < 0$ for at least one displacement field. The transition occurs when V_2 is positive but not definite. In that case there exists at least one displacement field for which V_2 vanishes, or

$$\min(V_2) = 0. \tag{2}$$

The discusser's equation (2) governs incipient buckling. Since in its variational form it is linear in the disturbed displacements, it represents the equations of equilibrium of a configuration which is contiguous to State I and which is associated with the same load. The customary adjacent equilibrium argument has thus been established. However, only V_2, rather than the full expression for ΔV, is involved, and hence the actual existence of such an equilibrium configuration is, in general,[8] not to be inferred.

C. V. SMITH.[9] This paper presents an extremely interesting and informative comparison of the effect of linearized prebuckling deformations on the magnitudes of critical loads. It appears, however, that certain objections can be raised concerning statements such as, "...the usual adjacent equilibrium argument presented in the literature, according to which only the displacements are perturbed, is not applicable for the determination of the bifurcation pressures of the shallow arch." One might infer from this statement, taken by itself, that the usual argument would give wrong answers for the magnitudes of the critical loads for a shallow arch. (This would be just an inference because the authors do not make such a conclusion.) However, the analysis shows that such is not the case. Indeed, equation (92) states that $\bar{p}_0 = 0$ and, therefore, equation (93) is exactly what one would get by applying the classical adjacent equilibrium criterion.

Perhaps the authors object to the usual criterion because of the fact that "..., at the bifurcation point, the deformations are symmetrical and unique." However, this uniqueness of deformation at the bifurcation point is characteristic of all bifurcation problems. When the axial compressive load on a perfect simply supported column reaches π^2EI/L^2, the unique equilibrium configuration is straight. This is true even though, in the neighborhood of the bifurcation point, an infinitesimal measure of bending deformation requires a higher-order infinitesimal change in load for equilibrium.

There are, of course, several possible definitions for the critical load. The classical approach can be nonprecisely stated as follows. Assume that the system is in an equilibrium configuration. Fix the value of the load, and then impose a disturbance on the system. If every infinitesimal disturbance leads to motion of the system infinitesimally close to the equilibrium configuration, then the equilibrium state is stable. If there exists any infinitesimal disturbance which leads to motion finitely removed from the equilibrium configuration, then the equilibrium is unstable. Critical loads might then be loosely defined as the boundary points between stable and unstable regions on a load magnitude line. It can be shown mathematically that for a conservative system, these critical loads can always be determined from the classical adjacent equilibrium criterion, even though the equilibrium configuration is unique at the critical point.

In view of the preceding discussion, it would be much appreciated if the authors would further comment on their objections to the usual adjacent equilibrium argument as applied to shallow arches. Once again thanks are extended for a very useful paper.

The example of the arch, as indeed most examples of elastic buckling (such as plates, shells, etc.), can be invoked to illustrate this point. Stated and interpreted properly, however, the customary argument is confirmed rather than contradicted.

Additional References

3 Thompson, J. M. T., "The Estimation of the Critical Loads," *Journal of the Mechanics and Physics of Solids*, Vol. 15, 1967, p. 311.

4 Masur, E. F., and Schreyer, H. L., "A Second Approximation to the Problem of Elastic Instability," *Proceedings of the Symposium on the Theory of Shells (Donnell Anniversary Volume)*, University of Houston, Texas, 1967, pp. 231–254.

5 Pearson, C., "General Theory of Elastic Stability," *Quarterly of Applied Mathematics*, Vol. 14, 1956, pp. 133–144.

6 Hill, R., "On Uniqueness and Stability in the Theory of Finite Elastic Strain," *Journal of the Mechanics and Physics of Solids*, Vol. 5, 1957, pp. 229–241.

7 Masur, E. F., "On Tensor Rates in Continuum Mechanics," *Zeitschrift für Angewandte Mathematik und Physik*, Vol. 16, 1965, pp. 191–201.

8 Koiter, W. T., "On the Stability of Elastic Equilibrium," thesis, Delft, H. J. Paris, Amsterdam, 1945.

Authors' Closure[10]

It is well known that any stability problem can be analyzed by the "dynamic method." Because this method is usually cumber-

[7] This discussion is valid only in an averaging sense (relative to an appropriate norm) and does not include singular behavior.

[8] A common, and pedagogically unfortunate, exception is the case of the column under axial load. If the effect of very large deflections is disregarded, then $\Delta V = V_2$, and the column buckles under constant load.

[9] Associate Professor, School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, Ga.

[10] Because the contested idea was suggested by Professor Kerr and since Dr. Soifer is presently fully occupied with nonrelated mechanics problems, it was agreed by the authors that Professor Kerr would write the reply.

some, for elastic conservative systems subjected to static loads the "energy method" or the so called "adjacent equilibrium method" is usually preferred.

According to the energy method for an equilibrium configuration to be stable the corresponding total potential energy Π has to be a proper minimum with respect to the Π's of *all* kinematically admissible states (not necessarily equilibrium states). Thus the stability proof of a given equilibrium configuration consists of showing that the necessary and sufficient conditions for Π to be a weak minimum are satisfied at the considered equilibrium configuration. This problem is discussed, for simple functionals, in a number of texts on the calculus of variations [9, 10].[11] For the more complicated functionals encountered in stability analyses of elastic solids, this problem is discussed by Koiter [11, 12].

A system is defined to be in "neutral" equilibrium when $\delta^2\Pi = 0$ for nonzero perturbations. The condition $\delta^2\Pi = 0$ is satisfied by the nonzero solutions of the corresponding Jacobi equations. It should be noted that the Jacobi equations are in general identical with the so called "variational equations" which are obtained by perturbing only the *displacements* in the equilibrium equations and then by retaining only the linear terms in the perturbations. The condition for the existence of nonzero perturbations which satisfy $\delta^2\Pi = 0$ is then used as criteria for the determination of the "critical" loads at which neutral equilibrium takes place.

In the adjacent equilibrium method it is *assumed* that an equilibrium state becomes neutral if there exists, for the same loads, at least one adjacent *equilibrium* configuration [13–15]. This assumption is identical with the stipulated *procedure* for obtaining the variational equations just discussed. Hence, as long as the variational equations and the Jacobi equations are identical, the two methods obviously yield the same critical loads.

In the discussed paper, it is claimed, that the *previous assumption that at a bifurcation point there always exists an adjacent equilibrium configuration, when the loads are held constant, is incorrect,* although the method yields the correct critical loads. (This claim is not restricted to shallow arches; the arch was meant as an example.) This is the only point questioned by all three discussers. It is a point of fundamental importance since it affects the teaching of structures also on the elementary level, and therefore has an influence on the development of the intuition, related to stability problems, of future engineers.

The question of the existence of an adjacent equilibrium position was posed some time ago by Pearson (reference [8] of paper), who was concerned about a number of conceptual difficulties which arise in connection with the usual adjacent equilibrium argument. Pearson found that also for his general formulation of elastic solids, the Jacobi equations and the variational equations are identical, and then concluded that therefore, at neutral equilibrium, an adjacent equilibrium state, keeping the loads constant, always exists.

It appears to this writer, that by showing that the Jacobi equations and the variational equations are identical, Pearson proved that the two *methods* lead to an identical formulation for neutral equilibrium (as expected); however, this does not constitute a proof that at neutral equilibrium (for example, at a bifurcation point) an adjacent equilibrium configuration always exists when the loads are kept constant.

Professor Masur, one of the discussers, uses essentially Pearson's argument to prove the existence of an adjacent equilibrium state. Therefore, the foregoing statement also applies to his argument.

Professor Koiter's objection appears to be based on a misunderstanding. In the paper it is *not* claimed that the "adjacent equilibrium" argument is not applicable because it may lead to incorrect bifurcation pressures, but rather that the argument is *physically* incorrect as an *equilibrium* argument. It is of interest

[11] Numbers in brackets designate additional References at end of Closure.

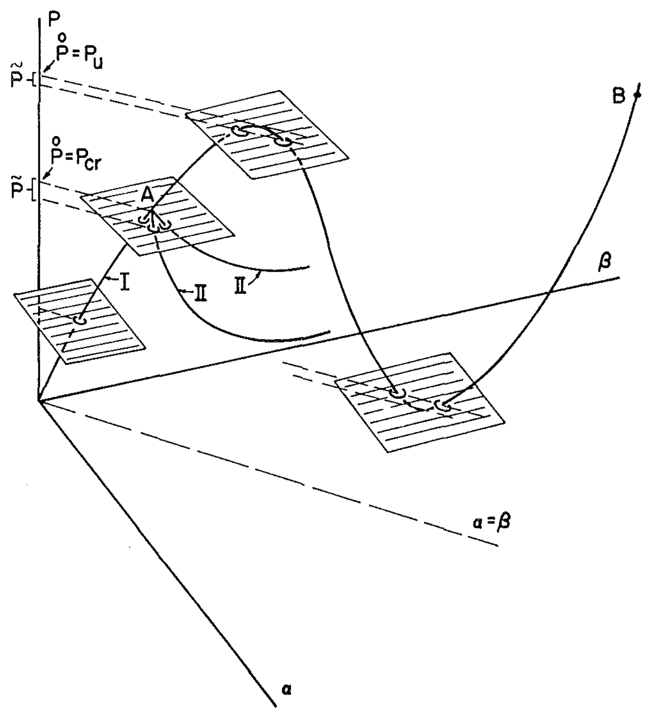

Fig. 1 (Note: branches II meet with the symmetric branch at point B)

to note that Professor Koiter's "mathematical reason for the validity of the usual *adjacent equilibrium* procedure in the determination of a bifurcation point" does not imply the existence of an adjacent equilibrium configuration. It does imply, however, that the argument on which the procedure used in the paper is based is physically correct.

The foregoing comments answer also some of the questions raised by Professor Smith.

To clarify some of the questions related to the perturbation of the load, let us consider a class of elastic conservative systems described by two degrees of freedom (for example, various variations of the symmetrical problem shown in Fig. 1 of paper). We assume that equilibrium of each system is described by the two nonlinear algebraic equations

$$\begin{aligned} f(\alpha, \beta, P) &= 0 \\ g(\alpha, \beta, P) &= 0 \end{aligned} \qquad (1)$$

which are equivalent to equations (4) and (5) for the system shown in Fig. 1 of paper. Their exact solution yields equilibrium branches, as shown in Fig. 1 of this Closure. Denoting by (°) the variables at the critical points, the equilibrium equations in (1), for each of these points, become

$$\begin{aligned} f(\mathring{\alpha}, \mathring{\beta}, \mathring{P}) &= 0 \\ g(\mathring{\alpha}, \mathring{\beta}, \mathring{P}) &= 0 \end{aligned} \qquad (2)$$

In the adjacent equilibrium method used in the literature it is *assumed* that there always exists an adjacent position of equilibrium for

$$P = \mathring{P}; \qquad \alpha = \mathring{\alpha} + \tilde{\alpha}; \quad \beta = \mathring{\beta} + \tilde{\beta}$$

which is described, in view of (1), by the equations

$$\begin{aligned} f(\mathring{\alpha} + \tilde{\alpha}, \mathring{\beta} + \tilde{\beta}, \mathring{P}) &= 0 \\ g(\mathring{\alpha} + \tilde{\alpha}, \mathring{\beta} + \tilde{\beta}, \mathring{P}) &= 0 \end{aligned} \qquad (3)$$

Expanding the equations in (3), using Taylor's theorem, and retaining only linear terms in (˜), we obtain, noting (2),

$$\left(\frac{\partial f}{\partial \alpha}\right)_0 \tilde{\alpha} + \left(\frac{\partial f}{\partial \beta}\right)_0 \tilde{\beta} = 0$$
$$\left(\frac{\partial g}{\partial \alpha}\right)_0 \tilde{\alpha} + \left(\frac{\partial g}{\partial \beta}\right)_0 \tilde{\beta} = 0 \qquad (4)$$

The equations in (4) constitute the usually encountered eigenvalue problem. The stability criterion, equivalent to the condition for the existence of a nontrivial solution $(\tilde{\alpha}, \tilde{\beta})$, is

$$\begin{vmatrix} \left(\frac{\partial f}{\partial \alpha}\right)_0 & \left(\frac{\partial f}{\partial \beta}\right)_0 \\ \left(\frac{\partial g}{\partial \alpha}\right)_0 & \left(\frac{\partial g}{\partial \beta}\right)_0 \end{vmatrix} = 0 \qquad (5)$$

At this point it should be noted that point $(\mathring{\alpha} + \tilde{\alpha}, \mathring{\beta} + \tilde{\beta}, \mathring{P})$, for any admissible nonzero $\tilde{\alpha}$ and for $\tilde{\beta}$, is in general not a point on any of the equilibrium branches near point A, and hence the equations in (3) are *not* equilibrium equations for such points.

Let us now consider, as a counterexample, the same problem using the method presented in the paper. *This method is based on the observation, that at all points which are of interest for a stability analysis* (as well as at point B which is not of interest) *there exists, in a close vicinity on a neighboring α-β-plane, more than one state of equilibrium* (which may or may not be stable). This is shown in Fig. 1. The equations of equilibrium for such points are those given in (2). The points of interest are those, for which on the plane $P = \mathring{P} + \tilde{P}$ there exists, *in a close vicinity,* more than one equilibrium configuration with the coordinates

$$\alpha = \mathring{\alpha} + \tilde{\alpha}; \quad \beta = \mathring{\beta} + \tilde{\beta}$$

Substituting these perturbed quantities into the equilibrium equations (1), we obtain

$$f(\mathring{\alpha} + \tilde{\alpha}, \mathring{\beta} + \tilde{\beta}, \mathring{P} + \tilde{P}) = 0$$
$$g(\mathring{\alpha} + \tilde{\alpha}, \mathring{\beta} + \tilde{\beta}, \mathring{P} + \tilde{P}) = 0 \qquad (6)$$

Expanding the foregoing equations using Taylor's theorem, and retaining only linear terms in (˜), we obtain, noting (2),

$$\left(\frac{\partial f}{\partial \alpha}\right)_0 \tilde{\alpha} + \left(\frac{\partial f}{\partial \beta}\right)_0 \tilde{\beta} = -\left(\frac{\partial f}{\partial P}\right)_0 \tilde{P}$$
$$\left(\frac{\partial g}{\partial \alpha}\right)_0 \tilde{\alpha} + \left(\frac{\partial g}{\partial \beta}\right)_0 \tilde{\beta} = -\left(\frac{\partial g}{\partial P}\right)_0 \tilde{P} \qquad (7)$$

Note that the equations in (6) are equilibrium equations.

Let us estimate the order of $\tilde{P}$ in terms of $\tilde{\alpha}$ and $\tilde{\beta}$. Because at the *limit points* the equilibrium branches are differentiable, we may write

$$\mathring{P} + \tilde{P} = P(\mathring{\alpha} + \tilde{\alpha}, \mathring{\beta} + \tilde{\beta}) = P(\mathring{\alpha}, \mathring{\beta}) + \left(\frac{\partial P}{\partial \alpha}\right)_0 \tilde{\alpha} + \left(\frac{\partial P}{\partial \beta}\right)_0 \tilde{\beta} + \frac{1}{2}\left[\left(\frac{\partial^2 P}{\partial \alpha^2}\right)_0 \tilde{\alpha}^2 + 2\left(\frac{\partial^2 P}{\partial \alpha\, \partial \beta}\right)_0 \tilde{\alpha}\tilde{\beta} + \left(\frac{\partial^2 P}{\partial \beta^2}\right)_0 \tilde{\beta}^2\right] + \ldots \qquad (8)$$

Noting that $P(\mathring{\alpha}, \mathring{\beta}) = \mathring{P}$ and that at the limit points

$$\left(\frac{\partial P}{\partial \alpha}\right)_0 = 0 \qquad \left(\frac{\partial P}{\partial \beta}\right)_0 = 0 \qquad (9)$$

it follows that at the limit points

$$\tilde{P} = \frac{1}{2}\left[\left(\frac{\partial^2 P}{\partial \alpha^2}\right)_0 \tilde{\alpha}^2 + 2\left(\frac{\partial^2 P}{\partial \alpha\, \partial \beta}\right)_0 \tilde{\alpha}\tilde{\beta} + \left(\frac{\partial^2 P}{\partial \beta^2}\right)_0 \tilde{\beta}^2\right] + \ldots \qquad (10)$$

thus of order higher than one in $\tilde{\alpha}$ and $\tilde{\beta}$.

Retaining in equations (7) only terms of the order of $\tilde{\alpha}$ and $\tilde{\beta}$, it follows that for the *limit points* the equations in (7) become homogeneous, as shown in (4). However at the *bifurcation point*, $\tilde{P}$ may be of the order of $\tilde{\alpha}$ and $\tilde{\beta}$ and hence, for these cases, the linearized equations in (˜) are those shown in (7).

The condition for the existence of more than one state of equilibrium in a close vicinity on the $\mathring{P} + \tilde{P}$ plane, is equivalent to the condition for the existence of a nontrivial $(\tilde{\alpha}, \tilde{\beta})$ at the limit points and to the condition for the existence of a nonunique $(\tilde{\alpha}, \tilde{\beta})$ at the bifurcation point A (and point B). For both cases this condition is

$$\begin{vmatrix} \left(\frac{\partial f}{\partial \alpha}\right)_0 & \left(\frac{\partial f}{\partial \beta}\right)_0 \\ \left(\frac{\partial g}{\partial \alpha}\right)_0 & \left(\frac{\partial g}{\partial \beta}\right)_0 \end{vmatrix} = 0 \qquad (11)$$

except that in the second case also an orthogonality condition has to be satisfied in order to insure the existence of a solution [16]. Condition (11) is identical with condition (5). Hence, as expected, both methods lead to the same results. However, the physical arguments are very different.

Because the argument used in the so-called adjacent equilibrium method is identical to the one prescribed for the determination of the variational equations, its justification is obviously derived from the energy criterion. Because it is not associated with a *true* adjacent equilibrium configuration, there is no reason to classify it as an equilibrium method. On the other hand, the method discussed in the paper *is* based on true equilibrium states.

It appears to this writer, that a reclassification of the adjacent equilibrium method as a special case of the energy criterion and the adoption of the method used in the paper as the equilibrium method will greatly contribute toward the elimination of the conceptual difficulties and misunderstandings which are reported in the literature (see Pearson's paper page 134 and Bolotin [15], page 43). This will also make the presentation of the elementary theory compatible with the physical intuition of engineering and science students.

As to the first part of Professor Masur's discussion, it should be noted that the first four paragraphs are essentially a restatement of a part of the Introduction. Professor Masur's statement, that for prebuckling deformations to have a significant effect, an arch has to be "shallow," is not correct. The most obvious counterexample is the *symmetric* in-plane buckling of a semicircular thin arch subjected to a load at the vertex. For recent results on this subject, the reader is referred to a paper by Huddleston [17].

While writing the paper the authors were not aware of the papers by Thompson (Masur's reference [3]) and by Masur and Schreyer (Masur's reference [4]) which appeared in 1967. A review of the reference by Masur and Schreyer revealed, however, that their presentation of the shallow arch and the presentation contained in the paper are very different, although the obtained stability loads, according to Masur, are the same.

From a historic point of view, it should be noted that as early as 1947, Friedrichs [18] succeeded in reducing the two simultaneous nonlinear differential equations for the shallow arch to one linear equation. A similar approach was used by Kornishin and Mushtari [19] in 1955 for the analysis of a shallow cylindrical strip. Schreyer and Masur (paper reference [22]) also used this procedure. Their solution is not complete in the sense that they did not evaluate the resulting formulation for the arch deflections, but for an average deflection w^*. The solution given in the paper is complete in that it is presented in terms of loads and arch displacements. The reader should note that the resulting exact solution also exhibits equilibrium branches not discussed in the previously stated papers. These branches are discussed in reference [20].

Thanks are extended to the discussers, Professors Koiter, Masur, and Smith, for their comments. It is hoped that their comments and the foregoing reply will contribute to the clarification of a fundamental question in the theory of stability of conservative elastic solids.

References

9 Bolza, O., "Vorlesungen über Variationsrechnung," B. G. Teubner, Leipzig und Berlin, 1909.

10 Gelfand, I. M., and Fomin, S. V., *Calculus of Variations*, Prentice Hall, 1963, Chapter 5.

11 Koiter, W. T., "Over de stabiliteit van het elastisch evenwicht" dissertation, Delft, H. J. Paris, Amsterdam, 1945.

12 Koiter, W. T., "Stability of Equilibrium of Continuous Bodies," Report, Division of Applied Mathematics, Brown University, Apr. 1962.

13 Timoshenko, S. P., and Gere, J. M., *Theory of Elastic Stability*, 2nd ed., McGraw-Hill, New York, 1961.

14 Pflüger, A., *Stabilitätsprobleme der Elastostatik*, 2nd ed., Springer, Berlin, 1964, p. 59.

15 Bolotin, V. V., *Nonconservative Problems of the Theory of Elastic Stability*, Pergamon Press, 1963.

16 Hildebrand, F. B., *Methods of Applied Mathematics*, Prentice Hall, 1952, p. 29.

17 Huddleston, J. V., "Finite Deflections and Snap-Through of High Circular Arches," Journal of Applied Mechanics, Vol. 35, No. 4, Trans. ASME, Vol. 90, Series E, Dec. 1968, pp. 763–769.

18 Friedrichs, K. O., "Seminar on Nonlinear Elasticity," New York University, 1947–1948, Section IX, Printed Notes.

19 Kornishin, M. S., and Mushtari, Kh. M., "The Instability of an Infinitely Long Shallow Cylindrical Strip Subjected to a Normal Uniform Pressure" (in Russian), *Izvestia Kazanskogo Filiala AN USSR, Seria fiz.-mat. i tekhn. nauk*, Nr. 7, 1955, pp. 36–50; see also Chapter XI of "Nonlinear Theory of Thin Elastic Shells," Mushtari, Kh. M., and Galimov, K. Z., Tatknigoizdat Kazan 1957. English translation available as NASA-TT-62, 1961.

20 Kerr, A. D., and El-Bayoumy, L., "On the Nonunique Equilibrium States of a Shallow Arch," *Quarterly of Applied Mathematics*, 1970.

An Asymptotic Solution for Laminar Flow of an Incompressible Fluid Between Rotating Disks[1]

M. DeSANTIS,[2] L. GALOWIN,[2] and E. RAKOWSKY.[2] The authors' solution for this problem illustrates the viscous dissipation of angular momentum in the radial inflow between parallel disks and is of special interest since we are currently investigating a similar phenomenon.[3] Results which display very similar behavior to the theoretical values of the radial velocity profiles were obtained in our experiments for $\Omega = 0$ with $h/r_0 = 0.031$. Development of nonsimilar velocity profiles from "fully developed" is anticipated in the presence of a varying strong favorable pressure gradient, e.g., see comments by Savage, reference [1].[4]

The flow field in a sink vortex device was investigated with application to a flueric angular rate sensor and was reported in reference [2]. In that study the development of the radial inflow of the fluid between parallel coaxial disks into an axisymmetric outlet was determined. Experimental velocity profiles were measured by hot-wire anemometry, and numerical solution of the axisymmetric Euler equations was obtained from a computer program. Comparisons of the radial velocity component, taken from Fig. 3,[1] with our experimentally measured values indicate substantially the same development with decreasing r, as shown in Fig. 1. Differences are attributed to the zero angular rate in the experiments.

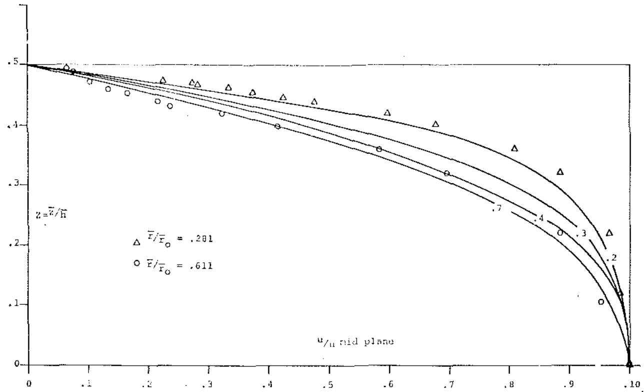

Fig. 1 Velocity profiles

Numerical results obtained from the inviscid analysis for $r \to 0$ indicate local maxima in the radial velocity component, i.e., inflections do occur in the profiles approaching the axis of symmetry.

In our case the pressure field is a function of r and z so that the adverse pressure gradient in the vicinity of the stagnation point decelerates the flow. Outside that region the flow is turned and accelerated into the outlet. In the analysis presented in the subject paper, $p(r)$ only is assumed, but with the varying centrifugal field (due to the z distribution of the v^2/r term becoming increasingly large at small z and decreasing r) the radial inflow is then subjected to an effective $p(r, z)$. A mechanism then becomes apparent for radial decelerations about $z = 0$ for decreasing r and inflections develop in the profile. A mechanism for acceptance of profile solutions with velocity overshoot is postulated and discussed in a Note on the Falkner-Skan equation by Libby and Liu, see reference [3]. Consequently, the interpretation that inflections in the radial velocity distributions are necessarily associated with the onset of transition to turbulence is speculative and unwarranted.

References

1 Savage, S. B., "Laminar Flow Between Parallel Plates," Journal of Applied Mechanics, Vol. 31, No. 4, Trans. ASME, Vol. 88, Series E, Dec. 1964, pp. 594–596.

2 Rakowsky, E. L., Galowin, L. S., and DeSantis, M. J., "Vortex Phenomena Study Continuation Program," final report GPSI 1590-6707, Aerospace Research Center, General Precision Inc., Little Falls, N. J., Dec. 1967.

3 Libby, P. A., and Liu, T. M., "Further Solutions of the Falkner-Skan Equation," *American Institute of Aeronautics and Astronautics Journal*, Vol. 5, 1967, pp. 1040–1042.

Authors' Closure

The authors appreciate the references to experimental work in the subject area dealt with analytically in the paper. The discussion concerning the meaning of inflected profiles is welcomed by the authors; in particular, the conclusion reached by the discussers is also held by the authors at the time of preparation of this closure.

[1] By Matsch, L., and Rice, W., published in the March, 1968, issue of the Journal of Applied Mechanics, Vol. 35, Trans. ASME, Vol. 90, Series E, pp. 155–159.

[2] Aerospace Research Center, Fluidics Department, General Precision Inc., Little Falls, N. J.

[3] Study of Vortex Phenomena performed under Contract DA-01-021-AMC-11203(Z), Guidance and Control Lab., USAMC, Redstone Arsenal, Huntsville, Ala.

[4] Numbers in brackets designate References at end of Discussion.