inflow cases. For no radial flow, the entire cavity becomes what might be termed a single toroidal vortex cell, and this situation persists for small negative values of the radial Reynolds number. However, this phenomenon disappears as the radial inflow is increased as can be seen in Fig. 7 of the paper.

Finally, we believe that the oscillatory secondary flows that appear in the "corners" between the inner cylinder and the end walls, as shown in Figs. 7 and 8, are indicative of the flow instabilities that Dr. Anderson asks about in his last question. The numerical calculations were stable for these conditions and the convergence was still good so that we ascribe this behavior to the physical situation and not to the numerical technique.

# On Evaluation of Natural Frequencies for a System of Equal Inertias and Equal Spring Stiffnesses[1]

**HANS BERGKVIST.**[2] The problem considered in the Brief Note can, using the same notation, be described by the difference equation

$$\alpha\delta^2 y_k + \mu y_k = 0$$

or

$$\alpha(y_{k+1} - 2y_k + y_{k-1}) + \mu y_k = 0 \qquad k = 1, 2 \ldots . n$$

under the boundary conditions

$$y_0 = y_{n+1} = 0$$

The characteristic values $\mu = \omega^2$ of this equation are known to be

$$\mu = \omega^2 = 4\alpha \sin^2 \left\{\frac{\lambda\pi}{2(1+n)}\right\} \qquad \lambda = 1, 2 \ldots . n$$

(compare, for example, Hildebrand[3]). This expression seems to be a simpler form of the result given in the paper discussed.

Thus the natural frequencies of the system are given by

$$\omega_\lambda = 2\sqrt{\alpha}\sin\frac{\lambda\pi}{2(1+n)} \qquad \lambda = 1, 2 \ldots . n \qquad (1)$$

Furthermore a one-term expansion of the sine will give a value of the lowest frequency that deviates from the exact one by less than 1 percent if $n \geqslant 6$; i.e.,

$$\omega_1 = 2\sqrt{\alpha}\sin\frac{\pi}{2(1+n)} \simeq \frac{\pi}{1+n}\sqrt{\alpha}$$

$$\text{if } \frac{\pi}{2(1+n)} \leqslant \frac{1}{4} \rightarrow n \geqslant 6$$

The illustrative example presented in the paper can be solved directly and exactly by expansion of the determinant and solution of the secular equation, thus:

$$\Delta_3 = \begin{vmatrix} 2\alpha - \omega^2 & -\alpha & 0 \\ -\alpha & 2\alpha - \omega^2 & -\alpha \\ 0 & -\alpha & 2\alpha - \omega^2 \end{vmatrix} = 0$$

$$= (2\alpha - \omega^2)\cdot\Delta_2 - \alpha^2\Delta_1 = (2\alpha - \omega^2)(\Delta_2 - \alpha^2)$$

which gives

$$\omega_1 = \sqrt{\alpha}\sqrt{2 - \sqrt{2}};\quad \omega_2 = \sqrt{\alpha}\sqrt{2};\quad \omega_3 = \sqrt{\alpha}\sqrt{2 + \sqrt{2}}$$

These values equal $2\sqrt{\alpha}\sin\frac{\pi}{8}$; $2\sqrt{\alpha}\sin\frac{\pi}{4}$, and $2\sqrt{\alpha}\sin\frac{3\pi}{8}$, respectively, which are also obtained from (1) for $n = 3$.

## Author's Closure

The writer thanks Mr. Bergkvist for writing this discussion. The writer will take this opportunity to quote two more references of importance [1, 2].[4] The eigenvalue equation (equation (3)) or its equivalent form such as the one mentioned by the discusser are derivable from many approaches. Besides Chebyshev's polynomial, previously mentioned, other methods include transfer matrix [1], Fibonacci number [2], as well as finite differences. Derivation by induction is an alternative straightforward method. The writer has applied mathematical induction to other systems [3].

Mr. Bergkvist has raised an interesting point regarding the approximation of the lowest eigenvalue using one-term expansion of the sine function. However, this may not be worthwhile, since the exact frequency equation is already simple enough for general purpose.

Finally, the writer might as well mention that the orthonormal eigenmodes associated with this problem (governed by equation (1) in the text) is [4]

$$\{q_\lambda\} = \sqrt{\frac{2}{1+n}}\left(\sin\frac{\lambda\pi}{1+n}, \sin\frac{2\lambda\pi}{1+n}, \ldots, \sin\frac{n\lambda\pi}{1+n}\right)^T$$

$$\lambda = 1, 2, \ldots, n$$

where $(\quad)^T$ represents the transpose.

More information on modeling and direct solution to a class of mechanical vibration systems with different kinds of boundary conditions are treated in a forthcoming paper [5].

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# On Integral Methods for Predicting Shear Layer Behavior[1]

**D. E. ABBOTT.**[2] This paper treats approximate solution techniques of the momentum integral type as applied to boundary-layer problems. Specifically, the author discusses the role that

[1] By Fan Y. Chen, published in the September, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 646–647.

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[3] Hildebrand, F. B., *Finite-Difference Equations and Simulations*, Prentice Hall, Englewood Cliffs, N. J., 1968, pp. 37–39.

[1] By S. J. Shamroth, published in the December, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 673–681.

[2] Professor, Purdue University, Fluid Mechanics Group, School of Mechanical Engineering, Lafayette, Ind.

is played by an assumed velocity profile in combination with the "momentum integral" method to provide an approximate solution of the boundary-layer equations as applied to a plane turbulent near wake. The author correctly points out that a mathematical singularity may arise with this method and associates the solution singularities found by Green [4], Reeves and Lees [5], and others as probably being related purely with the mathematics employed, rather than a physical singularity (although one never knows, a priori). The author then proceeds to discuss a specific application to the two-dimensional turbulent near wake and proposes three methods for circumventing the mathematical singularity and presents some numerical results showing the relative success of these alternate methods.

The purpose of this Discussion is to present some insight or perspective regarding the general solution technique as a whole, and thus to attempt to explain why the mathematical singularity arises (or may arise), and further to indicate how the singularity may be avoided altogether. The integral method itself belongs to the general approximation techniques categorized generally as the method of weighted residuals, or MWR (see [1–3], of the Discussion). The MWR contains two basic elements for its application, an approximating function for the dependent variable and a weighting function. The author discusses the role of the approximating function at some length (his "assumed velocity profile") and alludes to the role of the weighting functions (his "different sets of integral equations"), but does not strongly focus attention of the relationship between the two.

Mikhlin and Smolitskiy [4] show that the weighting functions and approximating functions may be thought of as operations which map the approximate solution onto the physical plane. Further, when the approximate solution lies wholly within the solution field defined by the weighting functions (that is, when they are identically orthogonal), the solution may be determined as accurately as desired. However, if the assumed approximating function is poorly chosen, then for a given weighting function, isolated singularities or lines of singularities will arise and poor results are obtained. Thus, in simple terms, it is easy to say that once a linearly independent set of approximating functions is chosen, a linearly independent set of weighting functions should exist if a physical solution in fact exists. How you would find such a set of weighting functions is something else.

In perspective, then, the author's three methods (switched equations, least-squares, and center-line equation), and in fact the original momentum integral equation, are just different assumptions for the weighting function, and sure enough, various degrees of success are obtained for each one. By relying on the concept of "subspace mapping" of Mikhlin and Smolitskiy, it is surely possible to do even better. It should *not* be implied by the author, however, that an a priori assumed approximating function will always lead to a singularity, nor that integral methods (as used loosely to label MWR techniques) always results in the difficulties described in the paper. On the other hand, if a particular integral type of method yields a suspicious looking singular behavior during the course of analysis, the mathematics should probably be seriously questioned before ascribing to any physical explanation.

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**IRWIN E. ALBER.**[3] Shamroth (see footnote one) has pointed out that a singularity can arise in the integral equations describing both boundary-layer flows at separation and wake flows at a point slightly downstream of the rear stagnation point. He then proceeds to prescribe several ad hoc methods, applied to the problem of the turbulent near wake, to avoid what he designates as a "velocity profile critical point."

It is the purpose of this Discussion to show ($a$) that the singularity noted by Shamroth is common to all forward-marching methods of solution of the boundary-layer equations, in the regions near separation and wake reattachment, whether finite difference or integral, when the external pressure, $P_e(x)$, is specified and ($b$) that the singularity can be eliminated by not prescribing the pressure, $P_e(x)$, but determining it *simultaneously* with the boundary-layer solution by matching the normal velocity induced by the growth of the boundary layer with that of the external inviscid flow.

Goldstein [5][4] has shown that a singularity appears at the separation point ($\tau_w = 0$) in the exact solution of the boundary-layer equations for incompressible flow with a prescribed linearly decreasing external velocity. Close to the separation point ($x = x_s$) $\tau_w \sim (x_s - x)^{1/2}$ and $v \rightarrow (x_s - x)^{-1/2}$ (except at $y = 0$). In fact, a singularity appears in the exact solution at the separation point for any prescribed pressure gradient. Goldstein has also shown this to be true at the stagnation point in the wake. Actual numerical finite-difference solutions of the boundary equations always "blow up" at or near the separation point when $P_e(x)$ is specified. Also, one can easily show that in a single parameter integral formulation, such as the one proposed by Tani [6], the determinant of the system of equations vanishes at the separation point.

It is generally agreed that real flows do not develop singularities at separation. In real flows near separation, the static pressure cannot be determined by an inviscid solution of the external flow for a given body shape, but is determined by the interaction between the boundary layer and the outer inviscid flow. This phenomenon is also true for reattaching wake flows as well, whether they be subsonic or supersonic.

Oswatitsch [7] has shown the regular nature of the flow around the separation point; the separating streamline angle being given by $(\tan\theta)_{\psi=0} = -3u_{xy}/u_{yy}$. At the rear stagnation point in the wake, a similar analysis gives the following expression for the dividing streamline trajectory

$$y_{\psi=0}{}^2 = -\frac{6u_x}{u_{yy}}x$$

By solving ($a$) the integral continuity equation, *simultaneously* with ($b$) the integral momentum and ($c$) the mean energy integral equation, so that the external velocity $u_e(x)$ is determined *simultaneously* along with the length scale, $\delta^*(x)$, and shape parameter, $a(x)$, Lees and Reeves [8] were able to integrate the compressible laminar boundary-layer integral equations through separation to reattachment for the shock wave boundary-layer interaction problem without encountering a singularity at separation. Their solution has been found to contain the essential features of the local Oswatitsch solution near separation and compares favorably with experimental data for a wide class of interacting boundary-layer flows at supersonic speeds. The determinant of the set of 3 simultaneous equations does not vanish at the separation point, and only vanishes at certain points in a supersonic flow problem where the flow on the average changes from subcritical $\int \frac{1 - M^2}{M^2}\,dy < 0$ to supercritical $\int \frac{1 - M^2}{M^2}\,dy > 0$.

[3] Principal Engineer, Northrop Corporate Labs, Hawthorne, Calif.

[4] Numbers in brackets designate Additional References at end of Discussion.

Similarly, no velocity profile critical point singularity is found in the wake solutions of Reeves and Lees [9] and Alber and Lees [10] when the full set of 3 equations are solved simultaneously; although a saddle point singularity exists in the supersonic flow problem which is analogous to the throat of a converging-diverging nozzle. This supersonic singularity is referred to as the "Crocco-Lees critical point."

So as not to confuse the two types of singularities just mentioned, we will illustrate the effect of (*a*) prescribing or (*b*) solving, simultaneously, for the external pressure distribution by considering the problem of the incompressible near wake (with splitter plate to avoid vortex shedding). For this problem, the flow is completely subcritical; thus any disturbance downstream can be propagated directly upstream to the base. Hence, for the *incompressible problem,* there can be *no* Crocco-Lees critical point. The integral momentum, mechanical energy, and continuity equations for the incompressible wake problem ($\tau_w = 0$) are given as[5]

$$\mathcal{H} \frac{d\delta^*}{dx} + \delta^* \frac{d\mathcal{H}}{dx} + [2\mathcal{H} + 1] \frac{\delta^*}{u_e} \frac{du_e}{dx} = 0 \qquad (1)$$

$$J \frac{d\delta^*}{dx} + \delta^* \frac{dI}{d\mathcal{H}} \frac{d\mathcal{H}}{dx} + 3J \frac{\delta^*}{u_e} \frac{du_e}{dx} = C_D = \frac{2}{\rho u_e^3} \int_0^\infty \tau \frac{\partial u}{\partial y} dy \qquad (2)$$

$$\frac{d\delta^*}{dx} - Z \frac{\delta^*}{u_e} \frac{du_e}{dx} = \tan \Theta \qquad (3)$$

The streamline angle $\tan \Theta$ can be related to the external velocity field by the equation of thin-airfoil theory as suggested by Green [11]; i.e.,

$$\tan \Theta(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x - \xi)[1 - u_e/u_\infty]}{(x - \xi)^2 + \delta^*} d\xi \qquad (4)$$

or conversely

$$\frac{u_e}{u_\infty}(x) = 1 + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x - \xi) \tan \Theta(\xi)}{(x - \xi)^2 + \delta^2} d\xi \qquad (5)$$

If one specifies the external velocity distribution, as did Shamroth[3] for the compressible wake problem and Green [11] for the incompressible wake problem, then one first solves equations (1) and (2) (for a specified family of velocity profiles and some related expression for $C_D$) to find $\delta^*(x)$ and $\Theta(x)$ in the region from the base to the far wake. Then $\delta^*$ and $\mathcal{H}$ are inserted into the continuity equation (3) to find $\tan \Theta(x)$. The calculated distribution of $\tan \Theta$ is next substituted into equation (5) and a new external velocity distribution $u_e(x)$ found. The whole procedure is then repeated with the new $u_e(x)$ until convergence is hopefully obtained. A basic problem arises in the integration of equations (1) and (2) with $u_e(x)$ given. The determinant of this system of equations is

$$\tilde{D} = \mathcal{H} \frac{dJ}{d\mathcal{H}} - J \qquad \text{(pressure given)} \qquad (6)$$

A plot of $\tilde{D}$ as a function of $\mathcal{H}$ (for the Stewartson family of velocity profiles) is shown in Fig. 1 (note $\mathcal{H} = 0.25$ at the rear stagnation point, and $\mathcal{H} \to 1$ as $x \to \infty$). Upstream of the rsp, $\tilde{D}$ is negative, but slightly downstream of the rsp, $\tilde{D}$ goes through zero at a point where $a = u_{\mathbb{C}}/u_e \approx 0.077$, and then becomes positive. It is this vanishing of the determinant when the pressure is specified that has led to the problem uncovered by Shamroth which he calls a velocity profile critical point. This same determinant ($\tilde{D}$) vanishes at the separation point for a wall boundary layer when $u_e(x)$ is given. Thus, in order not to get a singular solution, Green [11] was forced to adjust his pressure gradient at the $\tilde{D} = 0$ point so that the numerators, $N_1$ and $N_2$, in the expressions: $d\delta^*/dx = N_1/D$, $d\mathcal{H}/dx = N_2/D$ would vanish, thus insuring a regular solution. Another drawback to this method of solution is that $\tilde{D}$ is quite small in the region upstream of the rsp and any discrepancies in the assumed pressure gradient are greatly amplified in this region leading to unstable solutions.

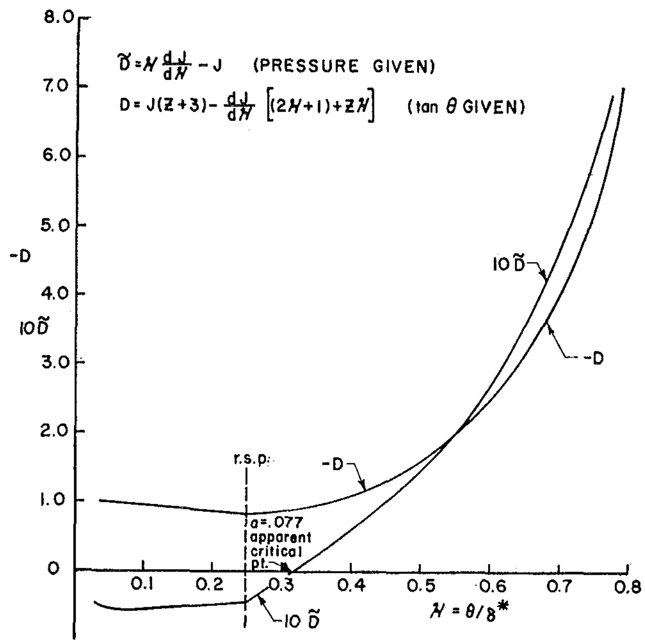


**Fig. 1 Determinant of integral equations system (Stewartson profiles), incompressible flow**

The proper way to avoid the velocity profile critical point is to solve equations (1)–(3) simultaneously, thus treating the flow as a strong interaction problem with the external velocity to be determined as part of the solution. The determinant of the full system of equations

$$D = J(Z + 3) - \frac{dI}{d\mathcal{H}} [(2\mathcal{H} + 1) + Z\mathcal{H}]$$

is also plotted against $\mathcal{H}$ in Fig. 1. One notes that $D$ is always negative and never goes through zero. Thus no singularity will be encountered in the integration of equations (1)–(3). To complete the problem, one should follow the steps outlined as follows:

1 Assume a distribution of $\tan \Theta(x)$, where $\tan \Theta \sim x^{-3/2}$ for large $x$.

2 Starting at the rsp, integrate equations (1)–(3) first downstream and then upstream of the rsp and match the solution with the base (see reference [13] for details).

3 Since the correct location of the rsp [i.e., $x(0)$] is not known a priori, different initial starting locations must be chosen until the wake solution satisfies the condition $a \to 1$, $u_e \to u_\infty$, as $x \to \infty$.

4 With the distribution of $u_e$ and $\delta$ calculated from equations (1)–(3), a new distribution of $\tan \Theta$ is determined from equation (5) and the whole process repeated.

For isentropic supersonic flow, $\tan \Theta$ is a local property (a function only of $M_e$ for Prandtl-Meyer flow) and thus it is not necessary to assume a distribution $\tan \Theta(x)$ as in the low-speed case.

Finally, it should be pointed out that regular solutions of the boundary-layer equations have also been obtained with finite-

[5] A full discussion of the incompressible turbulent wake problem is presented in the thesis by Alber [12], $\mathcal{H} = \frac{\theta}{\delta^*}$, $J = \frac{\theta^*}{\delta^*}$, $Z = \frac{\delta - \delta^*}{\delta^*}$.

difference numerical schemes past the point of vanishing skin friction (on a solid body) by calculating the pressure field as part of the solution. Two such solutions that might be cited are the incompressible laminar calculations of Catherall and Mangler [13] [$\delta_{(x)}{}^*$ is given in the separated zone; note tan $\Theta \sim d\delta^*/dx$] and the supersonic calculations of Tyson [14] [tan $\Theta = \nu(M_e) - \nu(M_\infty)$]. Both authors were able to obtain solutions with reverse flow and shallow separation bubbles. However, such methods are generally unstable in regions where the separation bubble is of the order of the upstream boundary layer. Thus, for those regions, some form of integral method is needed (at least below the $u = 0$ line) in order to assure a complete solution for a given separated flow field problem.

In summary, it should be evident that Shamroth's paper is incorrect on two major counts:

1 The velocity profile critical point is not a phenomenon solely attributable to integral methods, but is a basic singularity in the boundary-layer equations at points of zero shear and velocity, no matter what the method of solution, provided that the pressure distribution is specified a priori.

2 The velocity profile critical point can be correctly avoided (not by overdetermining the set of equations and using a least-squares method of solution) but by taking note of the fundamental character of strong interaction flows. Thus one determines the pressure distribution simultaneously with the rest of the boundary-layer solution, provided that tan $\Theta$ is either assumed and then iterated upon or is related directly to the local flow variables (as in supersonic flow).

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**J. E. GREEN.**[6] In view of the success with which integral methods have been applied to supersonic base flows, it is timely that Dr. Shamroth should draw attention to the difficulties which these methods may encounter as a result of a "velocity profile" singularity. However, while broadly accepting his exposition of the problem, the writer must take issue with his conclusion that, "...there are quite clear physical reasons for rejecting any constraint imposed by a velocity profile critical point and, therefore, the validity of any set of equations which encounters this singularity........must be questioned."

Dr. Shamroth rejects this singularity because there is a clash between its implications and those of the singularity which occurs at the station where $\int_0^\delta (1 - M^2/M^2)dy = 0$—the so-called Crocco and Lees critical point. In so doing, he places great weight on there being a physical significance in the concept of sub and supercritical flows as it follows from the model of Crocco and Lees [15].[7] The writer doubts the justification for this. On the contrary, he takes the view that sub and supercriticality are features peculiar to the model flow, and arise only because the assumption is made that static pressure is constant across the shear layer and is coupled to the pressure in the outer, inviscid stream through the flow direction at $y = \delta$. There are two main reasons for these doubts. First, experimental evidence tends to conflict with rather than support the concept of criticality embodied in the Crocco and Lees model; second, it appears possible to eliminate supercritical behavior by a refinement of the model. For example, according to the model the phenomenon of self-induced separation, in which the boundary layer generates a steeply rising pressure field by deflecting the external stream away from the wall, is possible only when the boundary layer is subcritical. But it is known from experiment (e.g., Bogdonoff and Kepler [16]) that a supersonic, turbulent boundary layer—which according to the model is supercritical and not capable of self-induced separation—does, in fact, separate by just such a process. Moreover—a crucial point—experiment indicates that during this process there are appreciable static pressure gradients across the boundary layer. Myring and Young [17] have recently shown how these transverse pressure gradients may be taken into account by describing the flow in terms of integrals along isobars. In a subsequent paper Myring [18] has shown that in this frame of reference, if isobars are assumed to be linear extensions of the outgoing characteristics of the external flow, a turbulent boundary layer in supersonic flow is "subcritical" and consequently, as he has shown, its self-induced separation can be predicted by an integral method.

Let us turn now to the question of how physically realistic is the constraint imposed by the velocity profile singularity. If we consider, for example, predicting the behavior of the viscous flow by simultaneous forward integration of the momentum and energy ($u$-moment of momentum) integral equations, the singularity occurs at the turning point in the ratio $H_{32}$ of the energy and momentum thicknesses. The fact that $dH_{32}/dx = 0$ at this point implies a balance between the dissipation integral and an appropriately scaled pressure gradient. Within the usual limits of boundary-layer theory (and, in fact, slightly beyond them, since the integral equations may be written in the reference frame of Myring and Young) this balance is exact in both the real and the model flows. At the position of minimum $H_{32}$ the discrepancy between a real and a computed flow—if they both satisfy the boundary-layer approximations—can arise only because the use of a one-parameter profile family in the computation results in approximate rather than exact evaluations of the dissipation integral and of the appropriate shape parameters. For laminar flows, however, a one-parameter profile family is capable of describing the interrelation of these various quantities at the singularity (as it occurs in a near wake) with good engineering accuracy. In constraining a solution to pass through this "profile" singularity, we are therefore merely forcing it to satisfy a condition

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[7] Numbers in brackets designate Additional References at end of Discussion.

which is a good approximation to the condition which will apply at the minimum in $H_{32}$ in the corresponding real flow. In turbulent flow there are considerably greater uncertainties in modeling the shear-stress field. For this reason the constraint imposed at, say, the position of minimum $H_{32}$ will be less realistic than in laminar flow, but then so, too, will be the solution throughout the wake, since errors in the turbulent shear-stress model will have overall rather than purely local consequences.

In integral treatments of a supersonic near wake, the difficulty which may arise, and to which Dr. Shamroth has drawn attention, is due to the occurrence of two saddle-point singularities rather than one. Crocco and Lees did not meet this difficulty because their assumed shape-parameter relationship, being empirically based and hence slightly oversimplified, did not exhibit a turning point; consequently no profile singularity occurred. The writer, in his treatment of a turbulent wake [19], did not meet it because he was concerned with subsonic flow, in which the Crocco and Lees singularity does not occur. In both cases, the occurrence of a single saddle-point singularity was of crucial importance, insuring uniqueness of solution in the near wake region. In the later treatments [20, 21] of a supersonic wake which Dr. Shamroth cites, a fairly realistic profile family was used; hence, as he observes, both the profile and the Crocco and Lees singularities might have been expected to occur. Perhaps the reason that no difficulties on this score were reported was that the two singularities lay in very close proximity. If this were so, the numerical technique used to extrapolate through the Crocco and Lees singularity might well have jumped across the profile singularity at the same time.

Whatever the real reason for the success of these methods, it seems fairly clear that, while the occurrence of one saddle point in the near wake is an apparently essential requirement for a satisfactory integral method, the occurrence of two should, in general, greatly increase the problem of obtaining a solution. In recognizing this, Dr. Shamroth has decided that of the two singularities it is the profile one which should be avoided. He describes three methods by which, accepting some (small) compromise in the treatment of the viscous flow, this may be achieved. On the other hand, the writer foresees the possibility that, by following Myring (and perhaps assuming a slightly different shape for the isobars, so that they remain realistic in regions where flow on the axis is subsonic), we might be able to eliminate the Crocco and Lees singularity. We should then be left with only one saddle point in the near wake, and at the same time should have actually improved the analysis by making a first-order allowance for transverse pressure gradients. But whether this approach would succeed as envisaged is at present entirely a matter for speculation.

To summarize, it appears expedient that integral treatments of the near wake should contain one, but only one, saddle-point singularity in the recompression region. However, we are not yet in a position to say what the precise nature of this singularity should be, and our grasp of the physical significance of the two types of singularity discussed here is far from satisfactory. It is the writer's view that, to be justified in categorically rejecting either of them *on physical grounds*, we should need a much better understanding of the local structure of the real flow than has yet been achieved.

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21 Alber, I. E., and Lees, L., "Integral Theory for Supersonic Turbulent Base Flows," *AIAA Journal*, Vol. 6, 1968.


**W. C. ROSE.**[8] Before discussing the content of the author's paper, I would like to thank him for sending me a copy with certain typographical errors corrected. Unfortunately, there are other errors not noted on that copy; they are:

1 Equation (4) is incorrect as it stands, and should read

$$\tau = \mu \frac{\partial u}{\partial y} - \overline{(\rho v)'u'} \text{ or } \tau = \mu \frac{\partial u}{\partial y} - \bar{\rho}\overline{u'v'}.$$

2 The Nomenclature of the superscripts is reversed.

3 The division sign is missing from the ordinate label in Fig. 1.

4 The term $A_{3,n} \dfrac{du}{dx}$ in equation (31) should read $A_{3,n} \dfrac{du_e}{dx}$.

5 The values of $m$ and $n$ are interchanged in the sentence before equation (24).

The discusser would like to thank the author for an informative presentation of the problems associated with the velocity profile critical point. The author presents three methods for coping with the problem: the method of switched equations, the method of center-line momentum equation, and the method of least squares. The discusser would like to comment on the first and third of the proposed methods; the author has commented adequately on the second method. The method of switched equations can enable one to obtain numerical results essentially uninfluenced by the critical point throughout the domain; however, if a global error estimate based on one system of equations exists, then that estimate is no longer valid when the system of equations is altered (i.e., the projection space—in the Hilbert space context—is altered). The use of the method of least squares proposed by the author is novel and provides reasonable predictions. The least-squares method explicitly involves $N$ equations; however, the author fails to point out what values of $N$ were used in obtaining the presented predictions.

The previous three methods could be used if a critical point were encountered but there are important types of global approximate solutions which do not encounter velocity profile critical points. The condition that determines if a critical point may be encountered, clearly depends on whether the matrix elements $a_{ij}$ of equation (11) are all independent of the $x$ coordinate. If the matrix elements are independent of $x$ (and) the matrix is invertible at the initial $x$-station (and, consequently, invertible everywhere) (then) no critical point will be encountered. In a paper by Murphy and Rose,[9] for example, an approximate solution was sought by first transforming the equations to Crocco variables (i.e., $(x, y) \rightarrow (x, u(x, y))$ and letting the approximation to the resulting unknown, $\partial u/\partial y$, be of the form

$$\left(\frac{\partial u}{\partial y}\right)_p = \sum_{i=1}^{p} \alpha_i(x)\phi_i(u),$$

where $\phi_i(u)$ is a sequence of linearly independent functions satisfying the boundary conditions. When the coefficients, which are analogous to the $a_{ij}$, are formed, they are all independent of $x$. The discusser does not propose this type of approximation as a "cure-all" for the velocity profile critical point problem, but

[8] NASA-Ames Research Center.

[9] Murphy, J. D., and Rose, W. C., "Application of the Method of Integral Relations to the Calculation of Incompressible Turbulent Boundary Layers," *Proceedings of the Computation of Turbulent Boundary Layers*, AFOSR-IFP-Stanford Conference, Aug. 1968.

rather that the search for such a function be considered along with the methods suggested by the author.

In the event that an approximation which is liable to encounter a velocity profile critical point is employed in a supersonic wake problem, the relation of the velocity profile critical point to that of the well-known Crocco-Lees point is of interest to this discusser. First, the author says, "The velocity profile critical point leads to a second constraint [the Crocco-Lees point] . . ." This seems to imply that a velocity profile critical point is required in order to have a Crocco-Lees point. It is not clear just what the suggested relation between the two points is. Further, it is not obvious that the critical point in the author's example (discussed in Figs. 1–5) is a profile critical point and not, in fact, a Crocco-Lees point. The author seems to suggest two criteria for determining whether a critical point is velocity profile critical point or a Crocco-Lees point. One is monitoring the value of $A$ in equation (15). When $A$ is zero, the author assumes the existence of a Crocco-Lees point. However, the Crocco-Lees point is always defined as the $x$-station where

$$\frac{d\delta}{dp} = \frac{d\delta}{dx}\left(\frac{dp}{dx}\right)^{-1} = 0,$$

and it has not been shown that this is the station where $A$ is zero. The other criteria seems to be whether the location of the critical point can be changed by changing the form of the integral equations (as in the method of switched equations). Since the location of the Crocco-Lees point can also be changed by this method, it is not clear that this could be used to distinguish them. Perhaps the author would comment on these (or other) methods he has used to assess whether a critical point is due to the velocity profile assumption or the Crocco-Lees-type singularity.

One final comment concerns the section entitled, "Physical Significance of Velocity Profile Critical Point." The author states that the principal objection to accepting a solution constrained by a velocity critical point is that if the equation were solved ". . . numerically without the introduction of a profile family" a critical point would not arise. Since nearly all numerical, or finite-difference, schemes can be shown to result from the assumption of some polynomial smoothing function over a subinterval of the domain of integration, these polynomials might, themselves, encounter critical points. Therefore, the fact that the equations might be solved by some arbitrary finite-difference procedure does not in itself insure one that no critical point can be encountered. The discusser feels that the principal objection to accepting solutions which exhibit a velocity profile critical point is not that some other approximate solution may not encounter one, but rather, as the author clearly points out earlier, it arises from mathematical considerations alone and not from the assumed governing differential equations.

## Author's Closure

The author wishes to thank the discussers for their interesting and informative comments, many of which shed light upon the nature of all singularities occurring in the near-wake recompression region.

The major portion of Dr. Alber's discussion simply demonstrates that although it is impossible to solve the set of boundary-layer integral equations at a velocity profile critical point, it is possible to solve the strong-interaction set of equations at a velocity profile critical point; the strong-interaction set of equations being the original set of boundary-layer integral equations plus an additional equation used to determine the streamwise pressure gradient. Since this same conclusion was stated explicitly in the original paper, no disagreement between Dr. Alber and the author exists concerning this conclusion. However, as previously stated by the author, when a solution to the strong-interaction set of equations is obtained at a velocity profile critical point, the predicted streamwise pressure gradient is constrained in a physically unrealistic manner, and therefore, the resulting solution should be viewed with skepticism. Of the several reasons presented for rejecting the constrained solution, two are particularly relevant to Dr. Alber's comments.

A major reason for rejecting the constrained solution is based upon the statement that no singularity analogous to a velocity profile critical point emerges in a solution of the full set of partial differential equations. Dr. Alber takes issue with this statement and claims that the velocity profile critical point is analogous to the well-known separation point singularity appearing in solutions of the partial differential boundary-layer equations. This possibility has been previously investigated by the author, who, after careful consideration, found no reason for assuming any analogy between the two singularities. On the contrary, there are several reasons for rejecting the suggestion that such an analogy exists. In the first place, both the results of the present turbulent calculation, which use a displaced Gaussian velocity profile, and the laminar calculations of Reeves and Lees (reference [5] of the original paper) which use a set of Stewartson profiles show a velocity profile critical point appearing in the attached flow well downstream of the wake rear stagnation point. For example, in the present calculations, the shape factor at the rear stagnation point is approximately 7.1 and the shape factor at the velocity profile critical point is approximately 5.8. The velocity profile critical point also appears in reattaching boundary layers using the method of McDonald [22][10] (which should replace reference [8] of the original paper). In these calculations which are based upon Coles' velocity profile family, a velocity profile critical point appears at a streamwise station approximately two boundary-layer thicknesses downstream of reattachment where the shape factor is approximately 3 as opposed to a separation value of 4.1. Thus the velocity profile critical point emerges in a region distinctly and consistently downstream of the rear stagnation point in these widely varying problems.

A second reason for rejecting the suggested analogy is found in the analysis of Stewartson as described by Brown and Stewartson [23]. Stewartson's analysis indicates that the stagnation point singularity does not occur in the presence of heat transfer in a compressible boundary layer; however, in an integral formulation if an approximate temperature-velocity relation is assumed, the velocity profile critical point (for a given ratio of wall to free-stream temperature) is only a consequence of the velocity profile family and does not disappear in the presence of heat transfer. Thus heat transfer suppresses the stagnation point singularity but does not suppress the velocity profile critical point. The final and most important reason for rejecting the suggested analogy lies in the origin of the separation singularity. The separation singularity emerges because the solution of the boundary-layer partial differential equations is required to satisfy a higher-order boundary condition [23] which is obtained by twice differentiating the streamwise momentum equation with respect to $y$ and then setting $y = 0$. As a result of this boundary condition, the pressure gradient at separation cannot be arbitrarily assigned. However, although a solution of the boundary-layer partial differential equations is required to satisfy this higher-order boundary condition, solutions of the integral equations need not, and in many instances cannot, satisfy such boundary conditions. In fact, solutions of the integral equations often do not even satisfy the first-order boundary condition equating the shear stress gradient at the wall and the streamwise pressure gradient. Since the mechanism giving rise to the separation singularity is absent from the integral equations used by this author, it does not seem reasonable to assume that an analogy between the velocity profile critical point and the separation singularity exists. If despite the lack of evidence one chooses to assume that an analogy exists, the removal of the velocity profile critical point is still justified. Since the Navier-Stokes equations contain no separation singularity [23], the separation singularity which appears in the boundary-layer approximation to the Navier-Stokes equations should be removed if at all possible. Thus, in conclusion, even if the

[10] Numbers in brackets designate Additional References at end of paper.

velocity profile critical point were analogous to the separation singularity, it should be removed.

An additional reason for rejecting the velocity profile critical point as a physically realistic constraint (which was not explicitly offered as a reason in the original paper) concerns the unrealistic subcritical-supercritical transition in the neighborhood of a velocity profile critical point. As stated in the original paper, a boundary layer may respond to perturbations in the streamwise pressure distribution in a subcritical or supercritical manner. By analogy with one-dimensional compressible gas flow, the outer edge flow angle of a subcritical boundary layer increases as the pressure gradient becomes more adverse and that of a supercritical boundary layer decreases as the pressure gradient becomes more adverse. Furthermore, Weinbaum (reference [9] of the original paper) has shown that when normal pressure gradients are neglected, the criticality of a boundary layer can be determined by evaluating $\int_0^\delta (M^2 - 1)/M^2 dy$. In a solution of a strong-interaction set of equations such as that discussed by Dr. Alber, equations (1)–(3) of Dr. Alber's discussion, it is important that the boundary layer respond correctly to changes in the streamwise pressure gradient since this response is used to calculate the value of the streamwise pressure gradient itself. However, in the vicinity of a velocity profile critical point, a boundary layer which should respond in a subcritical manner according to the sign of $\int_0^\delta (M^2 - 1)/M^2 dy$ responds incorrectly in a supercritical manner or vice-versa. Thus, in the vicinity of a velocity profile critical point, even the qualitative nature of the predicted boundary-layer response is incorrect and, therefore, since this response is instrumental in predicting the streamwise pressure distribution, the validity of the resulting streamwise pressure distribution and subsequent shear layer development must be viewed with caution.

In regard to Dr. Green's comments, Dr. Green and the author are in closer agreement than it may initially appear. A portion of the apparent disagreement stems from the use of the word "physical" in describing the criticality of a shear layer. In the original paper, only integral solutions of the boundary-layer equations which assume normal pressure gradients to be negligible are examined and, thus, when a boundary layer is said to be subcritical from a physical basis, it is assumed that no normal pressure gradients are present. Under this assumption the physical criticality can be calculated from Weinbaum's integral. The author agrees with Dr. Green's contention that the criticality of a shear layer can be changed by the presence of normal pressure gradients; a concept which has been demonstrated by Holden [24] and discussed at length by Shamroth and McDonald [25]. However, within the framework of Crocco-Lees type theories, which neglect normal pressure gradients, the clash between implications arising from the mathematical subcritical-supercritical transition at a velocity profile critical point and the nature of the criticality according to Weinbaum's integral must be faced and the resulting inconsistencies must be eliminated before the results of such a calculation can be accepted with confidence. The key point is that, in some vicinity of the velocity profile critical point, the integral solution will predict the boundary layer to have one criticality whereas, if a solution of the partial differential equations were initiated at this station using velocity and temperature profiles of the integral solution as initial conditions, the resulting solution of the partial differential equations will show the boundary layer to have the opposite criticality. Therefore, in the framework of existing Crocco-Lees theories, an unacceptable inconsistency emerges and, at the present time, there is no reason to doubt the validity of the partial differential equations in the immediate vicinity of the station at which a velocity profile critical point occurs in an integral solution.

It should be repeated that although the apparent clash between the velocity profile critical point and the Crocco-Lees critical point is one reason for rejecting the physical reality of the velocity profile critical point, it is not the most compelling reason. The most compelling reason for rejection is that no such singularity appears in the solution of the governing partial differential equations.

The author disagrees with Dr. Green in regard to the contention that the occurrence of one saddle point in the near wake is an apparently essential requirement for a satisfactory integral method. Based upon an analysis of a linearized strong-interaction set of equations, Weinbaum and Garvine [26] concluded that the appearance of a saddle-point singularity (the Crocco-Lees critical point) is a result of improperly treating the strong-interaction problem as an initial value problem rather than properly treating the strong-interaction problem as a boundary-value problem. The validity of this conclusion has been demonstrated by Shamroth and McDonald [25] who treat the interaction problem as a boundary-value problem and, in addition, avoid a velocity profile critical point by using the "method of least squares." The results of reference [25] show that when the near-wake problem is properly posed as a boundary-value problem no singularity emerges.

The author appreciates the comments of Dr. Abbott which show how the solution techniques proposed in the original paper fit into the method of weighted residuals. It is certainly possible that viewing the problem from this frame of reference may lead to different and perhaps better methods of avoiding the velocity profile critical point.

The author would like to thank Dr. Rose for pointing out several typographical errors in the paper. The first two errors mentioned were corrected before publication. In the results presented in the original paper $N$ is set equal to 17. Concerning Dr. Rose's comment on the Crocco-Lees point it should be pointed out that there is no relation between the velocity profile critical point and the Crocco-Lees critical point. Weinbaum (reference [9] of the original paper) has demonstrated from an analysis of the boundary-layer partial differential equations that the Crocco-Lees critical point occurs at the streamwise station at which the boundary-layer outer-edge flow angle becomes insensitive to small changes in the streamwise pressure gradient rather than at the streamwise station at which $d\delta/dp = 0$. In terms of the equations presented in the original paper, an examination of equation (21) of the original paper shows that the streamwise station suggested by Weinbaum at which the outer-edge flow angle becomes insensitive to the pressure gradient is also the station at which the determinant of the strong-interaction set of equations obtained from equations (16)–(18) of the original paper

$$\begin{vmatrix} A_1 & A_2 & -A_3 \\ B_1 & B_2 & -B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

is equal to zero. Thus the location of this critical point as given by Weinbaum is identical to the location given by Lees and Reeves (reference [5] of the original paper) and Alber and Lees (reference [6] of the original paper). In a strong-interaction solution based upon the boundary-layer partial differential equations, the station at which the outer-edge flow becomes insensitive to the streamwise pressure gradient coincides with the station at which $A = 0$. However, in a solution based upon a set of integral equations, the two stations do not coincide; although, in general, they are quite close together. This lack of coincidence is due to the approximate nature of the integral solutions. In integral solutions the location of the Crocco-Lees critical point can be determined by monitoring the sensitivity of the outer-edge flow angle to small changes in the pressure gradient, or monitoring the determinant of the strong-interaction set of equations.

The velocity profile critical point occurs when the determinant of coefficients of the governing integral equations (excluding the strong-interaction equation) is zero (see equation (13) of the original paper). The velocity profile critical point and the Crocco-Lees critical point can occur simultaneously only if the determinant of coefficients of the set of integral equations goes to zero at the same station at which the determinant of the strong-interaction set of equations goes to zero. In terms of the example presented in equations (16)–(18) of the original paper, this would require

$$\begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} A_1 & A_2 & -A_3 \\ B_1 & B_2 & -B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

to be zero simultaneously. In general, such an occurrence is highly unlikely.

In addition to the difference in definitions, it should be noted that a velocity profile critical point can appear in incompressible flow (see Green reference [4] of the original paper) whereas, a Crocco-Lees critical point does not occur in incompressible flow. Furthermore, by definition, no velocity profile critical point can appear in an analytic solution of a strong-interaction set of equations based upon the boundary-layer partial differential equations, but a Crocco-Lees critical point may be encountered. Finally, according to Weinbaum (reference [9] of the original paper) at the Crocco-Lees critical point the outer-edge flow angle becomes insensitive to the streamwise pressure gradient; however, at the velocity profile critical point (for a two-parameter profile family) equation (22) of the original paper shows that the outer-edge flow angle becomes extremely sensitive to the streamwise pressure gradient unless the two singularities coincide.

## Addtional References

22 McDonald, H., and Camarata, F. J., "An Extended Mixing-Length Approach for Computing the Turbulent Boundary-Layer Development," *Proceedings of the Computation of Turbulent Boundary Layers*, AFOSR-IFP-Stanford Conference, Aug. 1968.

23 Brown, S. N., and Stewartson, K., "Laminar Separation," *Annual Review of Fluid Mechanics*, Vol. 1, 1969.

24 Holden, M. S., "Theoretical and Experimental Studies of the Shock Wave-Boundary Layer Interaction on Curved Compression Surfaces," *Symposium on Viscous Interaction Phenomena in Supersonic and Hypersonic Flow*, Hypersonic Research Laboratory (OAR), Wright-Patterson Air Force Base, Ohio, 1969.

25 Shamroth, S. J., and McDonald, H., "A New Solution of the Turbulent Near-Wake Recompression Problem," AIAA Paper No. 70-228, 1970.

26 Weinbaum, S., and Garvine, R. W., "On the Two-Dimensional Viscous Counterpart of the One-Dimensional Sonic Throat," *Journal of Fluid Mechanics*, Vol. 39, 1969.