inflow cases. For no radial flow, the entire cavity becomes what might be termed a single toroidal vortex cell, and this situation persists for small negative values of the radial Reynolds number. However, this phenomenon disappears as the radial inflow is increased as can be seen in Fig. 7 of the paper.

Finally, we believe that the oscillatory secondary flows that appear in the "corners" between the inner cylinder and the end walls, as shown in Figs. 7 and 8, are indicative of the flow instabilities that Dr. Anderson asks about in his last question. The numerical calculations were stable for these conditions and the convergence was still good so that we ascribe this behavior to the physical situation and not to the numerical technique.

On Evaluation of Natural Frequencies for a System of Equal Inertias and Equal Spring Stiffnesses[1]

HANS BERGKVIST.[2] The problem considered in the Brief Note can, using the same notation, be described by the difference equation

$$\alpha\delta^2 y_k + \mu y_k = 0$$

or

$$\alpha(y_{k+1} - 2y_k + y_{k-1}) + \mu y_k = 0 \qquad k = 1, 2 \ldots . n$$

under the boundary conditions

$$y_0 = y_{n+1} = 0$$

The characteristic values $\mu = \omega^2$ of this equation are known to be

$$\mu = \omega^2 = 4\alpha \sin^2 \left\{\frac{\lambda\pi}{2(1+n)}\right\} \qquad \lambda = 1, 2 \ldots . n$$

(compare, for example, Hildebrand[3]). This expression seems to be a simpler form of the result given in the paper discussed.

Thus the natural frequencies of the system are given by

$$\omega_\lambda = 2\sqrt{\alpha}\sin\frac{\lambda\pi}{2(1+n)} \qquad \lambda = 1, 2 \ldots . n \qquad (1)$$

Furthermore a one-term expansion of the sine will give a value of the lowest frequency that deviates from the exact one by less than 1 percent if $n \geqslant 6$; i.e.,

$$\omega_1 = 2\sqrt{\alpha}\sin\frac{\pi}{2(1+n)} \simeq \frac{\pi}{1+n}\sqrt{\alpha}$$

$$\text{if } \frac{\pi}{2(1+n)} \leqslant \frac{1}{4} \rightarrow n \geqslant 6$$

The illustrative example presented in the paper can be solved directly and exactly by expansion of the determinant and solution of the secular equation, thus:

$$\Delta_3 = \begin{vmatrix} 2\alpha - \omega^2 & -\alpha & 0 \\ -\alpha & 2\alpha - \omega^2 & -\alpha \\ 0 & -\alpha & 2\alpha - \omega^2 \end{vmatrix} = 0$$

$$= (2\alpha - \omega^2)\cdot\Delta_2 - \alpha^2\Delta_1 = (2\alpha - \omega^2)(\Delta_2 - \alpha^2)$$

which gives

$$\omega_1 = \sqrt{\alpha}\sqrt{2 - \sqrt{2}}; \quad \omega_2 = \sqrt{\alpha}\sqrt{2}; \quad \omega_3 = \sqrt{\alpha}\sqrt{2 + \sqrt{2}}$$

These values equal $2\sqrt{\alpha}\sin\frac{\pi}{8}$; $2\sqrt{\alpha}\sin\frac{\pi}{4}$, and $2\sqrt{\alpha}\sin\frac{3\pi}{8}$, respectively, which are also obtained from (1) for $n = 3$.

[1] By Fan Y. Chen, published in the September, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 646–647.

[2] Research engineer, Division of Strength of Materials, Lund Institute of Technology, S-22007 Lund, Sweden.

[3] Hildebrand, F. B., *Finite-Difference Equations and Simulations*, Prentice Hall, Englewood Cliffs, N. J., 1968, pp. 37–39.

Author's Closure

The writer thanks Mr. Bergkvist for writing this discussion. The writer will take this opportunity to quote two more references of importance [1, 2].[4] The eigenvalue equation (equation (3)) or its equivalent form such as the one mentioned by the discusser are derivable from many approaches. Besides Chebyshev's polynomial, previously mentioned, other methods include transfer matrix [1], Fibonacci number [2], as well as finite differences. Derivation by induction is an alternative straightforward method. The writer has applied mathematical induction to other systems [3].

Mr. Bergkvist has raised an interesting point regarding the approximation of the lowest eigenvalue using one-term expansion of the sine function. However, this may not be worthwhile, since the exact frequency equation is already simple enough for general purpose.

Finally, the writer might as well mention that the orthonormal eigenmodes associated with this problem (governed by equation (1) in the text) is [4]

$$\{q_\lambda\} = \sqrt{\frac{2}{1+n}}\left(\sin\frac{\lambda\pi}{1+n}, \sin\frac{2\lambda\pi}{1+n}, \ldots, \sin\frac{n\lambda\pi}{1+n}\right)^T$$

$$\lambda = 1, 2, \ldots, n$$

where $(\quad)^T$ represents the transpose.

More information on modeling and direct solution to a class of mechanical vibration systems with different kinds of boundary conditions are treated in a forthcoming paper [5].

References

1 Pipes, L. A., "The Matrix Theory of Torsional Oscillations," *Journal of Applied Physics*, Vol. 13, July 1942, pp. 434–444.

2 Rutherford, D. E., "Some Continuant Determinants Arising in Physics and Chemistry," *Proceedings, Royal Society of Edinburgh*, Series A, Vol. 62, 1945, pp. 229–236; also, Series A, Vol. 63, 1951, pp. 232–241.

3 Chen, F. Y., "On Degeneracy of Eigenvalues and Recursive Solution of Symmetrically Coupled Dynamic Systems," to be published in the JOURNAL OF APPLIED MECHANICS.

4 Haynesworth, E. V., "Applications of a Theory on Partitioned Matrices," *Journal of Research of National Bureau of Standards*, Series B, Vol. 62, No. 2, 1959, pp. 73–78.

5 Chen, F. Y., "In Modeling and Direct Solution of Certain Free Vibration Systems," to be published in *Journal of Sound and Vibration*.

[4] Numbers in brackets designate References at end of Closure.

On Integral Methods for Predicting Shear Layer Behavior[1]

D. E. ABBOTT.[2] This paper treats approximate solution techniques of the momentum integral type as applied to boundary-layer problems. Specifically, the author discusses the role that

[1] By S. J. Shamroth, published in the December, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 673–681.

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