almost all experimental investigations of viscoelastic beam vibration. Since relatively little work dealing with the effect of imperfections in viscoelastic beams has been published, this paper is a valuable contribution to the field.

Author's Closure

The author would like to thank Professor Wiley and Dr. Francis for their kind remarks.

Dr. Francis' comments with regard to the response for $\omega \simeq 2\Omega$ are well taken. As was indicated in the paper (at the end of the section on Analytical Work), parametric resonance for $\omega \simeq 2\Omega$ is a distinct possibility. This possibility was investigated in reference [11] for the case of a perfect column. References [5, 6] show that imperfections have little effect on the response in the vicinity of this resonance, so no further analysis was presented.

The subharmonic response for $\omega \simeq 2\Omega$ is the dominant instability for the perfect viscoelastic column. However, in the presence of imperfections, responses at other frequencies can also be significant. One of the objectives of this paper was to show that this is so. In no way was it intended to imply that possible parametric resonance in the first instability region could be disregarded.

A Theoretical and Experimental Study of Confined Vortex Flow[1]

O. L. ANDERSON.[2] The theoretical analysis and experimental data presented in this paper represent a valuable contribution to our understanding of confined vortex flow. The potential benefits which may accrue with the development of a vortex reactor are so great that continued research in this area is needed. This paper invites several comments and like most research prompts many new questions.

1 It is very rare that the theoretical analyst and experimental investigator examine exactly the same problem. This paper accomplishes this difficult task to our benefit and the agreement is satisfying.

2 Much of the early theoretical work, authors' references [7–12], were accomplished only after many simplifications and assumptions had been made. It is gratifying that the exact numerical analysis presented in this paper produces results in general agreement with these earlier analysis. Thus the basic assumptions have been confirmed.

3 Although the equations are singular at the axis of symmetry and thus difficult to solve numerically, solutions obtained by removing the inner porous boundary would prove very valuable. Could the authors comment on this possibility?

4 Have these numerical solutions predicted "vortex cell" flow of the type predicted in the authors' reference [11] and examined experimentally by Travers.[3] The solutions for very small $N_{Re,r}$ appear to indicate this possibility.

5 These vortex flows are known to have instabilities. Occasionally exact numerical solutions predict instabilities. Is there any evidence in your calculations that this has occurred?

T. S. CHANG[4] **and C. W. KITCHENS, JR.**[5] The authors have presented a most interesting numerical and experimental study of a Newtonian vortex confined between rotating porous concentric cylinders. The geometry considered is somewhat different from that depicted in Fig. 1 of the paper; however, the numerical solution exhibits much of the phenomena in the confined vortex that has been observed experimentally. The numerical technique employed in this study should have application in many other related problems where it is desirable to work with the full Navier-Stokes equations.

It will be interesting to extend this technique to the solution of the non-Newtonian confined vortex. A study of rotating flows of non-Newtonian fluids has recently been completed[6] using the constitutive equation for a second-order fluid considered by Coleman and Noll [1][7]

$$\sigma_{ij} = -P\delta_{ij} + \eta_0 \overset{(1)}{A}_{ij} + \beta \overset{(1)}{A}_i{}^k \overset{(1)}{A}_{kj} + \gamma \overset{(2)}{A}_{ij},$$

where σ_{ij} is the stress tensor, P is the pressure, $\overset{(1)}{A}_{ij}$ and $\overset{(2)}{A}_{ij}$ are the first two Rivlin-Ericksen tensors [2], η_0 is the viscosity, and β and γ are normal stress coefficients. The case of the motion near a stationary wall, when the non-Newtonian fluid far from the wall rotates at a constant angular velocity, is closely related to the flow near the end wall of a confined vortex. Fig. 1 in this Discussion shows the radial and tangential velocity distribution functions (F and G) for such non-Newtonian flows, using the notation of Bödewadt [3], where K and L are nondimensional parameters related to β and γ, respectively. It appears that the difficulties associated with flow being diverted into the boundary layers in the non-Newtonian confined vortex may be even more pronounced than in the Newtonian case.

[1] By G. J. Farris, G. J. Kidd, Jr., D. W. Lick, and R. E. Textor, published in the December, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 687–692.

[2] Senior Research Engineer, Theoretical Gas Dynamics, United Aircraft Research Laboratories, United Aircraft Corporation, East Hartford, Conn.

[3] Travers, A., "Experimental Investigation of Radial Inflow Vortexes in Jet-Injection and Rotary-Peripheral-Wall Water Vortex Tubes," NASA CR-1028, Apr. 1968.

[4] NSF Professor, North Carolina State University, Raleigh, N. C.; also Cornell University, Ithaca, N. Y.

[5] First Lieutenant, U. S. Army, Ballistic Research Laboratories, Aberdeen, Md.

[6] Kitchens, C. W., Jr., "Vortex Flows of Second-Order Non-Newtonian Liquids," PhD thesis, North Carolina State University, Raleigh, N. C., 1970.

[7] Numbers in brackets designate References at end of this Discussion.

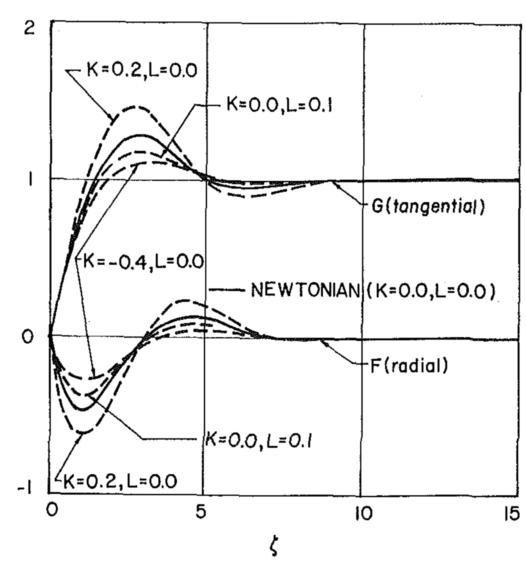

Fig. 1 Newtonian and non-Newtonian velocity distribution functions

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2 Rivlin, R. S., and Ericksen, J. L., "Stress Deformation Relations for Isotropic Materials," *Journal of Rational Mechanics and Analysis*, Vol. 4, No. 2, 1955, pp. 323–425.

3 Bödewadt, U. T., "Die Drehströmung über festem Grunde," *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 20, 1940, pp. 241–250.

C. G. RICHARDS.[8] The authors are to be commended for a very nice piece of work which undoubtedly involved a goodly amount of effort and skill.

However, this writer believes that the authors should now extend their present work by apparently slight—but significant—modifications of the boundary conditions.

The authors discuss two types of flows in their present paper. This discussion has been restricted to the second case.

For the second type of flow which the authors discuss (i.e., radial flow crosses the boundaries) the boundary condition at the inner cylinder is given as

$$\psi(r_i,z) = r_i v_{r,i} z \qquad 0 \leqq z \leqq z_l$$

This is a linear distribution which may not be realistic in that it requires the radial velocity to have no variation along the outlet.

From Figs. 7 and 8 of the paper, this restriction appears to have a rather drastic effect on the flow pattern (streamlines) in the region near the inner cylinder. The flow is essentially "forced" (numerically) to become uniformly distributed at the (inner cylinder) outlet. The question is, simply, "Does this accurately represent the flow in the present case"?

The reason for using such a condition on the inner boundary is that the computational procedure is greatly simplified. It may also be the difference between achieving or not achieving stability, although this is probably not the case here.

Perhaps a more realistic condition would be to require that the flow enter the inner cylinder horizontally, i.e.,

$$\frac{\partial\psi(r_i,z)}{\partial r} = 0$$

In this way, the location of the streamlines on the inner cylinder would not be fixed, but would be determined by the flow field. However, this condition is more difficult to apply in practice and may lead to other difficulties.

A second modification which might be considered would be to remove the inner cylinder from the problem and utilize a sink (i.e., drain hole) at the bottom surface of the chamber. This would more closely approximate the conceptual application depicted in the authors' Fig. 1. (In addition, the chamber might be made stationary with fluid being introduced tangentially as in the authors' Fig. 1.)

Removal of the inner cylinder would allow the region of interest to be extended to the axis of symmetry, where the motion is solid-body rotation with an unknown angular velocity. The sink might then be either a point sink (singularity) or a finite sink. The point sink would cause computational difficulties, while the finite sink would be very convenient—provided the velocity distribution was specified.

It is felt that perhaps these modifications may yield results which will give even more insight into the phenomenon being studied.

[8] Associate Professor, Department of Mechanical Engineering, The University of New Mexico, Albuquerque, N. Mex. Assoc. Mem. ASME.

NIKOLAUS ROTT.[9] As a comment to this valuable contribution to our knowledge of rotating flows, I merely wish to point out a few cases which would be worth investigating by the authors' numerical method, and which could lead to a critical assessment of analytical methods. For instance, it would be interesting to raise the tangential Reynolds number until even for the numerical method the special treatment of boundary-layer regions would become desirable; results could be compared to those obtained by the methods of references [6 and 11] of the paper. A different problem of great theoretical interest could be treated in the case of zero radial flow, again in the high Reynolds number limit, but letting the two end walls rotate with only a small velocity difference. The critical comparison with analytical results given elsewhere in the literature would be very valuable.

[9] Professor, Eidg Technische Hochschule, Institut für Aerodynamik, Zurich, Switzerland.

Authors' Closure

The authors would like to thank the reviewers for their comments and suggestions. As several of the reviewers point out, there are many additional aspects of this type of flow that should be investigated and we would hope to be able to carry on some of this work in the future. The potential usefulness of the vortex nuclear reactor, as well as the importance of vortex flows in other fields, are compelling reasons for further efforts in this area.

In responding to the reviews, we would like to discuss the various modifications mentioned first and then reply to the specific questions raised.

The elimination of the central exit tube, as mentioned by Dr. Anderson and Dr. Richards, would indeed lead to additional mathematical difficulty; however, we feel that this could be handled. The major problem is of course specifying the nature of the sink. The finite sink is of particular practical interest since it is the type that would probably be used in a reactor. The case in which the flow is totally enclosed, that is, in which there is no net radial flow and hence no need for a sink, has been studied by Pao, reference [14] of the original paper.

Dr. Richard's suggestion of the use of the boundary condition that the flow enter the central tube horizontally is another possible way of handling the sink, and would be interesting to try. However, in our experimental study, the resistance of the central tube was high enough to give essentially uniform flow, so that we feel the calculations do depict the actual physical situation.

The two extensions recommended by Professor Rott are excellent examples of "test cases" with which it should be possible to prove or disprove the validity of the use of simplications, such as momentum integral techniques, in the solution of this class of problems. A special case of the second type of flow he discusses is described in Fig. C3 of reference [18] of our paper. This is the case for no net radial flow, the cylindrical walls rotating, and both end walls held stationary. The tangential Reynolds number was relatively small (120) however. Our experience showed that computer times became unreasonably large for the high Reynolds number cases and some improvement in the computing technique would probably be needed to allow investigation of flows in this class.

The application of this technique to non-Newtonian flows, as discussed by Professor Chang and Mr. Kitchens, is yet another area of potential interest. The inclusions of a generalized stress tensor in the calculational scheme would open up a number of possible extensions, including, perhaps, the area of turbulent flows. As a final comment on extensions of this study, it should be mentioned that consideration of MHD effects would be of value in relation to studies of a plasma-vortex-reactor generator.

In answer to Dr. Anderson's question in point 4 of his discussion, a vortex cell type of flow does appear in the low radial inflow cases. For no radial flow, the entire cavity becomes what might be termed a single toroidal vortex cell, and this situation persists for small negative values of the radial Reynolds number. However, this phenomenon disappears as the radial inflow is increased as can be seen in Fig. 7 of the paper.

Finally, we believe that the oscillatory secondary flows that appear in the "corners" between the inner cylinder and the end walls, as shown in Figs. 7 and 8, are indicative of the flow instabilities that Dr. Anderson asks about in his last question. The numerical calculations were stable for these conditions and the convergence was still good so that we ascribe this behavior to the physical situation and not to the numerical technique.

On Evaluation of Natural Frequencies for a System of Equal Inertias and Equal Spring Stiffnesses[1]

HANS BERGKVIST.[2] The problem considered in the Brief Note can, using the same notation, be described by the difference equation

$$\alpha\delta^2 y_k + \mu y_k = 0$$

or

$$\alpha(y_{k+1} - 2y_k + y_{k-1}) + \mu y_k = 0 \qquad k = 1, 2 \ldots n$$

under the boundary conditions

$$y_0 = y_{n+1} = 0$$

The characteristic values $\mu = \omega^2$ of this equation are known to be

$$\mu = \omega^2 = 4\alpha \sin^2 \left\{\frac{\lambda\pi}{2(1+n)}\right\} \qquad \lambda = 1, 2 \ldots n$$

(compare, for example, Hildebrand[3]). This expression seems to be a simpler form of the result given in the paper discussed.

Thus the natural frequencies of the system are given by

$$\omega_\lambda = 2\sqrt{\alpha} \sin \frac{\lambda\pi}{2(1+n)} \qquad \lambda = 1, 2 \ldots n \tag{1}$$

Furthermore a one-term expansion of the sine will give a value of the lowest frequency that deviates from the exact one by less than 1 percent if $n \geqslant 6$; i.e.,

$$\omega_1 = 2\sqrt{\alpha} \sin \frac{\pi}{2(1+n)} \simeq \frac{\pi}{1+n}\sqrt{\alpha}$$

$$\text{if } \frac{\pi}{2(1+n)} \leqslant \frac{1}{4} \rightarrow n \geqslant 6$$

The illustrative example presented in the paper can be solved directly and exactly by expansion of the determinant and solution of the secular equation, thus:

$$\Delta_3 = \begin{vmatrix} 2\alpha - \omega^2 & -\alpha & 0 \\ -\alpha & 2\alpha - \omega^2 & -\alpha \\ 0 & -\alpha & 2\alpha - \omega^2 \end{vmatrix} = 0$$

$$= (2\alpha - \omega^2)\cdot\Delta_2 - \alpha^2\Delta_1 = (2\alpha - \omega^2)(\Delta_2 - \alpha^2)$$

which gives

$$\omega_1 = \sqrt{\alpha}\sqrt{2 - \sqrt{2}}; \quad \omega_2 = \sqrt{\alpha}\sqrt{2}; \quad \omega_3 = \sqrt{\alpha}\sqrt{2 + \sqrt{2}}$$

These values equal $2\sqrt{\alpha}\sin\frac{\pi}{8}$; $2\sqrt{\alpha}\sin\frac{\pi}{4}$, and $2\sqrt{\alpha}\sin\frac{3\pi}{8}$, respectively, which are also obtained from (1) for $n = 3$.

[1] By Fan Y. Chen, published in the September, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 646–647.

[2] Research engineer, Division of Strength of Materials, Lund Institute of Technology, S-22007 Lund, Sweden.

[3] Hildebrand, F. B., *Finite-Difference Equations and Simulations*, Prentice Hall, Englewood Cliffs, N. J., 1968, pp. 37–39.

Author's Closure

The writer thanks Mr. Bergkvist for writing this discussion. The writer will take this opportunity to quote two more references of importance [1, 2].[4] The eigenvalue equation (equation (3)) or its equivalent form such as the one mentioned by the discusser are derivable from many approaches. Besides Chebyshev's polynomial, previously mentioned, other methods include transfer matrix [1], Fibonacci number [2], as well as finite differences. Derivation by induction is an alternative straightforward method. The writer has applied mathematical induction to other systems [3].

Mr. Bergkvist has raised an interesting point regarding the approximation of the lowest eigenvalue using one-term expansion of the sine function. However, this may not be worthwhile, since the exact frequency equation is already simple enough for general purpose.

Finally, the writer might as well mention that the orthonormal eigenmodes associated with this problem (governed by equation (1) in the text) is [4]

$$\{q_\lambda\} = \sqrt{\frac{2}{1+n}}\left(\sin\frac{\lambda\pi}{1+n}, \sin\frac{2\lambda\pi}{1+n}, \ldots, \sin\frac{n\lambda\pi}{1+n}\right)^T$$

$$\lambda = 1, 2, \ldots, n$$

where $(\quad)^T$ represents the transpose.

More information on modeling and direct solution to a class of mechanical vibration systems with different kinds of boundary conditions are treated in a forthcoming paper [5].

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[4] Numbers in brackets designate References at end of Closure.

On Integral Methods for Predicting Shear Layer Behavior[1]

D. E. ABBOTT.[2] This paper treats approximate solution techniques of the momentum integral type as applied to boundary-layer problems. Specifically, the author discusses the role that

[1] By S. J. Shamroth, published in the December, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 673–681.

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