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discussion

Improved Linearized Velocity Profiles for Turbulent Free Shear Layers[1]

R. H. PAGE.[2] The authors are to be congratulated on their detailed analytical studies of alternate velocity distributions for use with integral conservation equations. Their analytical results must now be compared with experimental data. The determination of whether a Reichardt-type or a Göertler-type velocity profile is the truest representation must be based on the determination of which gives the best agreement with experimental data. Both profiles represent solutions of linearized equations and there is no reason *a priori* to believe that one linearization method is superior to the other.

The authors' statement that smaller values of η_m indicate a better profile may be misleading. Small values of η_m are necessary for the integral procedure but not sufficient for drawing conclusions about exactness or preciseness. For example, η_m for the similar solution of the zero secondary velocity case can be written as

$$\eta_m = \eta_R - \int_{-\infty}^{\eta_R} [\rho/\rho_\infty]\phi^2 d\eta \tag{1}$$

If we desire η_m to be zero (obviously the smallest possible value) this simply means that

$$\int_0^{\eta_R} d\eta = \int_{-\infty}^{\eta_R} [\rho/\rho_\infty]\phi^2 d\eta \tag{2}$$

which can be rewritten as

$$\int_0^{\eta_R} 0.5[1 - F(\eta)]d\eta = \int_{-\infty}^{0} 0.5[1 + F(\eta)]d\eta \tag{3}$$

where

$$[\rho/\rho_\infty]\phi^2 = 0.5[1 + F(\eta)] \tag{4}$$

Equation (3) will be satisfied when $F(\eta)$ is represented by any odd function, such as

$$F(\eta) = 1 - e^{-\alpha\eta^2} \tag{5}$$

$$F(\eta) = \sin \beta\eta \tag{6}$$

$$F(\eta) = \text{erf } \eta \tag{7}$$

Equation (7) represents Reichardt's velocity profile for the incompressible case. The authors have used this expression but it is obvious that equation (7) is only one of a number of possible profiles which satisfy the condition that η_m equals 0. Thus the fact that $\eta_m = 0$ merely means that $F(\eta)$ is an odd function and should not, by itself, be considered a test for exactness of the profile. In fact, the Göertler velocity profile, $\phi = 0.5(1 + \text{erf } \eta)$, for which η_m is not zero may be a better representation. Only a careful comparison of the theoretical profiles with experimental data can lead to a determination of which is a "better" profile.

[1] By J. P. Lamb, T. F. Greenwood, and J. L. Gaddis, published in the December, 1969, issue of the Journal of Applied Mechanics, Vol. 36, Trans. ASME, Vol. 91, Series E, pp. 657–663.

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Authors' Closure

The authors appreciate Professor Page's very interesting comments concerning the relative merits of the two types of linearized profiles. We agree with his statement that there is no reason, *a priori* to favor either type of profile. His arguments are concerned largely with the effect of linearization on profile shape. This is a valid inquiry which should be explored further, especially as regards the choice of a basic distribution, i.e., error, Gaussian, or trigonometric functions.

The primary thesis of our presentation was that, inasmuch as the type of linearization appears to have a relatively small effect on profile shape, Fig. 4, the location of the profile in space, as characterized by the integral shift parameter η_m, is of equal importance in determining the suitability of a given type of profile for use in an integral analysis. In this regard the Reichardt-type profiles were shown to be superior to the Oseen distributions. We did not mean to imply that, because of their lower η_m values, the former profiles were "better" in all respects; however, based on the foregoing reasoning, we did conclude that they were "better representations of the flow."

The question to be answered by an individual investigator, when contemplating the alternative profiles, is whether the simplicity afforded by elimination of the shift is offset by the possibility that the inaccuracy due to profile shape *might* be increased slightly. This will, of course, remain a judgment decision until further investigation is complete.

End Effect Bending Stresses in Cables[1]

ROBERT PLUNKETT.[2] The author presents a very neat demonstration of how the solution for the elastica can be modified to take care of the boundary conditions of the title problem for

[1] By J. A. DeRuntz, Jr., published in the December, 1969, issue of the Journal of Applied Mechanics, Vol. 36, Trans. ASME, Vol. 91, Series E, pp. 750–756.

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the case of constant tension. The case of the cable with non-negligible weight has a rather extensive bibliography for both the static [1, 2, 3][3] and the dynamic cases [4, 5, 6]. Eringen [7] shows that extensional deformations need not be neglected in deriving the author's equation (17). In reference [3], the methods developed by Moser [8] and Wasow [9] for singular perturbation differential equations have been used to obtain solutions of this equation similar to those of equations (41) and (42) of the author's paper but for the more general case of linearly varying axial tension caused by the weight of the cable. It is also shown there that the author's restriction to small angles is not necessary and that the effects of constant horizontal and vertical forces can be absorbed into the boundary conditions. The results he obtains, indicating that superposition can be used for calculating bending moments in the presence of lateral forces even for rather large angles, should be of interest for the important practical problem of offshore casings loaded laterally by tidal currents.

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3 Plunkett, R., "Static Buckling Stresses in Catenaries and Drill Strings," *Journal of Engineering for Industry*, TRANS. ASME, Series B, Vol. 89, 1967, pp. 31–36.

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(*b*) Frost, M. A., III, and Wilhoit, J. C., Jr., "Analysis of the Motion of Deep-Water Drill Strings—Part 2: Forced Rolling Motion," ibid., 1965, pp. 145–149.

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8 Moser, J., "Singular Perturbation of Eigenvalue Problems for Linear Differential Equations of Even Order," *Communications on Pure and Applied Mathematics*, Vol. 8, 1955, pp. 251–278.

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R. SCHMIDT[4] **and D. A. DaDEPPO.**[5] The nonlinear differential equations governing the three problems considered by the author can also be linearized with the aid of Chebyshev polynomials [10–12].[6] Such linearization yields more accurate results than the so-called "small-angle approximation," i.e., linearization with the aid of Maclaurin expansions.

Let us imagine the cable extended below the horizontal plane of fixity and take

$$\varphi = \gamma - \theta, \qquad -\gamma \leq \varphi \leq \gamma$$

The author's equation (18) then becomes

$$\varphi'' - \lambda^2 \sin \varphi = 0 \tag{1}$$

where primes denote derivatives with respect to s. According to [13] of this Discussion,

$$\sin \varphi \approx \frac{2}{\gamma} J_1(\gamma)\varphi, \qquad \cos \varphi \approx J_0(\gamma) \tag{2}$$

where J_0 and J_1 are the ordinary Bessel functions of the zeroth and first order, respectively. Substituting (2) into (1) and letting

$$k^2 = \frac{2}{\gamma} \lambda^2 J_1(\gamma) = \frac{2T^*}{\gamma EI} J_1(\gamma) \tag{3}$$

we obtain the approximate equation

$$\varphi'' - k^2\varphi = 0 \tag{4}$$

which is satisfied by

$$\varphi = C_1 \sinh (ks) + C_2 \cosh (ks) \tag{5}$$

From the conditions $\varphi = \gamma$ at $s = 0$; and $\varphi = 0$, $M = -EI\varphi' = 0$ at $s = L$,

$$C_1 = -\gamma \coth (kL) = -\gamma \tanh (kL), \qquad C_2 = \gamma \tag{6}$$

Hence,

$$\tanh (kL) = 1 \tag{7}$$

$$\varphi = \gamma[\cosh (ks) - \sinh (ks)] \tag{8}$$

$$M_0 = [2\gamma J_1(\gamma) EIT^*]^{1/2} \tag{9}$$

For example, for $\gamma = 2$ ($\gamma \approx 114.59$ deg), exact $M_0 = 1.683(EIT^*)^{1/2}$; from equation (9), $M_0 = 1.519(EIT^*)^{1/2}$; and by the small-angle approximation, $M_0 = 2(EIT^*)^{1/2}$. Of course, the accuracy of the method is much better for smaller angles.

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Author's Closure

The author would like to thank Professors Plunkett, Schmidt, and DaDeppo for their informative comments and the lists of additional references they have provided which include several related papers of which the author was unaware. In particular, it is gratifying to the author to know that some of the results of the paper have a wider range of application than indicated in the paper.

Flow in a Two-Dimensional Channel With a Rectangular Cavity[1]

ZEÉV ROTEM.[2] The authors are to be complimented on their investigation of this important problem which has a direct bearing

[3] Numbers in brackets designate References at end of Discussion.

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[6] Numbers in brackets designate Additional References at end of Discussion.

[1] By U. B. Mehta and Zalman Lavan, published in the December, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 897–901.

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