Incidentally, solutions for the form of area variation considered by the author, as well as many other forms, have been discussed in detail by Eisner.[4]

Author's Closure

The writer wishes to thank Professors Bert and Egle for pointing out an error in the paper. However, the solution remains valid for bars with variable cross section described by equation (1) in the paper.

[4] Eisner, E., "Complete Solutions of the 'Webster' Horn Equation," *Journal of the Acoustical Society of America*, Vol. 41, Apr. 1967, pp. 1126–1146.

Instability in an Elastic-Plastic Cylindrical Shell Under Axial Compression[1]

S. C. BATTERMAN.[2] Professors Ariaratnam and Dubey have indeed achieved the objective stated in the first sentence of the paper. It is valuable to obtain an expression for the critical bifurcation stress of an axially compressed cylindrical shell for a wide class of inelastic shell material response. The main purpose of this discussion, however, is to take exception with the conclusion that the results indicate that the bifurcation stress is highly sensitive to the variation in shape of the local yield surface.

It has been stated previously [1][3] that determinations of bifurcation values of stress using the Mises material model are representative of those that would be obtained by using any reasonable theory. Figs. 1 and 2 of the paper can be shown to put this statement on firm theoretical ground although the authors' interpretation of Figs. 1 and 2 leads to the conclusion that "... the critical stress is quite sensitive to the variation in the unit normal to the local yield surface." It is very unfair, and indeed erroneous, to compare values of S(or σ) as m_2 varies for the *same* value of tangent modulus. For a real material, where the tangent modulus is a function of stress, the system will not buckle at the same value of tangent modulus for different values of m_2. For a given idealization and a uniaxial stress-strain curve for an engineering material one should use the results of the analysis to compute a curve of σ versus t/a. Although differences will of course exist (except for $\nu = 1/2$) between bifurcation values of stress for different idealizations, the differences will be small and certainly nowhere near as large as implied by Fig. 2 for the Mises and Tresca solids. This is because small variations in stress lead to large variations in tangent modulus and will essentially wipe out the apparent large differences shown in Fig. 2. It is well known that it is precisely this effect which is responsible for the closeness of reduced modulus and tangent modulus predictions for columns although if the column formulas were plotted in the style of Fig. 2 they would appear to show enormous differences.

The discusser also notes that considerations essentially identical to those in the last section of the paper concerning the loading criterion have been presented previously [2]. Although it was unknown to Professors Ariaratnam and Dubey at the time they wrote their paper, it is also worth mentioning here that the precise specification of boundary conditions is a very significant factor in bringing the buckling predictions of incremental theory into better agreement with experiments [3]. In fact, this effect may be more important than considerations of the kind discussed in the paper.

[1] By S. T. Ariaratnam and R. N. Dubey, published in the March, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 47–50.

[2] Associate Professor of Engineering Mechanics, University of Pennsylvania, Philadelphia, Pa.

[3] Numbers in brackets designate References at end of Discussion.

References

1 Batterman, S. C., "Plastic Buckling of Axially Compressed Cylindrical Shells," *American Institute of Aeronautics and Astronautics Journal*, Vol. 3, Feb. 1965, p. 316.

2 Batterman, S. C., "Tangent Modulus Theory for Cylindrical Shells: Buckling Under Increasing Load," *International Journal of Solids and Structures*, Vol. 3, July 1967, p. 500.

3 Batterman, S. C., "Free-Edge Plastic Buckling of Axially Compressed Cylindrical Shells," JOURNAL OF APPLIED MECHANICS, Vol. 35, No. 1, TRANS. ASME, Vol. 90, Series E, Mar. 1968, pp. 73–79.

Authors' Closure

The authors note the objection of the discusser to our statement, "... the critical stress is quite sensitive to the variation in the unit normal to the local yield surface." It is true that the variation in bifurcation stress will not be large in the range of deformation in which the Young's modulus E and the tangent modulus E_t are of the same order of magnitude. However, the variation becomes significant in the range where plastic deformation is predominant or, more precisely, when $E_t \ll E$. For example, for bilinear solids, the large variation in critical stress for $\lambda > 10$ is apparent from Fig. 2 of our paper. A numerical prediction of the difference in the critical stress for general elastic-plastic solids can be made only if the general stress-strain law covering the entire range of deformation, from purely elastic deformation to predominantly plastic deformation, is known. Unfortunately, such a general stress-strain law is, to our knowledge, still lacking.

However, we did compute the value of the bifurcation stress for von Mises and for Tresca yield surfaces on the basis of information available in [4]. We approximated the stress-strain curve for the material quoted in [4] (Fig. 12), by $\sigma = A \exp \gamma$. This, together with equation (19) of our paper, was used to calculate the bifurcation stress. We were surprised to find that the values of 50,000 psi and 45,000 psi for σ_{crit} for the Tresca surface for $R/h = 50$ and 80, respectively, were almost the same as the experimental values we estimated from Fig. 16 of [4].

In addition, we would like to point out that Sewell [5, 6] and Dubey [7] also arrived at essentially the same conclusion, that is, "... the critical stress is quite sensitive to the variation in the unit normal ..."

We agree that the imposed boundary condition may have significant effect on the bifurcation stress. One of us has, in fact, discussed this effect in connection with a slightly different problem in the stability of cylindrical shells [7].

Additional References

4 Batterman, S. C., "Plastic Buckling of Axially Compressed Cylindrical Shells," Brown University report, 1964.

5 Sewell, M. J., "A General Theory of Elastic and Plastic Plate Failure—1," *Journal of the Mechanics and Physics of Solids*, Vol. 11, 1963, p. 377.

6 Sewell, M. J., "A General Theory of Elastic and Plastic Plate Failure—2," *Journal of the Mechanics and Physics of Solids*, Vol. 12, 1964, p. 279.

7 Dubey, R. N., "Instabilities in Thin Elastic Plastic Tubes," *International Journal of Solids and Structures*, Vol. 5, 1969, p. 699.