known at what point the solution becomes valid. The authors feel that to correctly describe the flow field in the wake, a method of solution treating the complete flow region must be used. We attempted to do this numerically and appear to have had some success.

In closing, we would like to point out two papers which study the flow in the trailing-edge region in some detail. Talke and Berger[7] use the method of series truncation as suggested by Van Dyke, and Stewartson[8] uses an improved modified Oseen-type analysis.

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[7] Talke, F., and Berger, S. A., "Analysis of the Laminar Incompressible Flow Near the Trailing Edge of a Flat Plate Using Series Truncation," paper presented at the annual meeting of the Division of Fluid Mechanics of APS, Seattle, Wash., Nov. 1968.

[8] Stewartson, K., "On the Flow Near the Trailing Edge of a Flat Plate," *Proceedings of the Royal Society*, Series A, Vol. 306, 1968, pp. 275–290.

The Strongest Circular Arch—A Perturbation Solution[1]

ROBERT SCHMIDT.[2] Professor Wu's paper is a valuable contribution to the store of knowledge of structural optimization. The perturbation technique used seems to yield useful results with a comparatively modest expenditure of effort. As mentioned by the author, a circular arch with clamped ends, which is subjected to a uniform external hydrostatic pressure, could also have been treated by the procedure described in the paper, though with greater difficulties. In the writer's opinion, hinged as well as clamped symmetrical arches under nonuniform loads could also be treated by the author's procedure, even though the classical theory of buckling of arches seems to be inapplicable for that kind of load because of large prebuckling deflections. However, for a practical range of values of the central angle 2α, the classical theory seems to predict buckling loads that are in fair agreement with exact results. Professor DaDeppo and the writer[3] have examined a very unfavorable case of load; namely, a concentrated load P at the crown of hinged-hinged circular arches. A semicircular arch under such a load buckles by sidesway at the time when the vertical deflection of the crown amounts to about 20 percent of the radius a. Certainly this is a large deflection. The exact value of the critical load, as calculated by means of elliptic integrals using the inextensional theory of circular elastica, is $5.8605\ EI/a^2$, as compared to $6.0EI/a^2$ obtained by the classical small-deflection theory of stability. Fair agreement also exists for arches that are shallower than the semicircular arch:

α	Exact Pa^2/EI	Classic Pa^2/EI
30 deg	22.66	23.45
40 deg	17.15	17.17
50 deg	13.82	13.33
60 deg	11.44	10.69
70 deg	9.46	8.74
80 deg	7.62	7.22
90 deg	5.86	6.00
100 deg	4.29	4.98

The reason for this unexpected agreement is obscure. To consider it as mere coincidence is difficult, since the similarity of values extends over a wide range. Most likely the agreement is due to a high value of fairly constant prebuckling thrust.

As seen from the author's Fig. 1, the calculated depth of the "strongest" arch rib vanishes at the two hinged supports and at the crown; i.e., at points at which the bending couples are zero. To transmit the thrust, the depth must not be zero. For that reason, it might have been better to have used the expression $h_c + h(x)$ to describe the depth of the arch rib, together with the extensional theory of arches.

The weight of the arch frequently is not an insignificant factor. In future researches, it should be included in the equations. In view of the foregoing discussion, inclusion of the weight might not invalidate the usefulness of the classical eigenvalue theory of stability for certain values of the central angle.

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[1] By C.-H. Wu, published in the September, 1968, issue of the Journal of Applied Mechanics, Vol. 35, Trans. ASME, Vol. 90, Series E, pp. 476–480.

[2] Professor of Engineering Mechanics, University of Detroit, Detroit, Mich.

[3] Schmidt, R., and DaDeppo, D. A., "On Buckling of Deep Arches at Large Deflections," submitted for publication in the *AIAA Journal*.

Author's Closure

The author thanks Professor Schmidt for his kind comments.

The method of perturbation is a powerful tool for the determination of the solution of a problem containing a small parameter ϵ for which the reduced problem ($\epsilon = 0$) exists and has a simple solution. The reduced problem for the case of a circular arch under uniform pressure is a fictitious column under axial load. It is not clear to the author that such a reduced problem exists for the case of a circular arch under arbitrary load.

Wave Propagation in a Finite-Length Bar With a Variable Cross Section[1]

C. W. BERT[2] **and D. M. EGLE.**[3] The writers object to the author's claim that the solution of the author's equation (2) for an arbitrary variation in cross section of the form

$$A = A_0 \sum_{n=1}^{\infty} a_n (x/\mu l)^n$$

may be obtained by superimposing the solutions obtained for each term in the series. The fallacy of this claim may be shown by considering the following two-term series as an example:

$$A = A_0(x + x^3)$$

If u_1 and u_3 are the solutions of the author's equation (3) with $n = 1, 3$, respectively, then the author's claim is that

$$u = u_1 + bu_3 \tag{1}$$

where b is a constant, is a solution of

$$\frac{\partial}{\partial x}\left[(x + x^3)\frac{\partial u}{\partial x}\right] = \frac{1}{c^2}(x + x^3)\frac{\partial^2 u}{\partial t^2} \tag{2}$$

Substitution of (1) into (2) will show that this can be true only if

$$x^2 \frac{\partial u_1}{\partial x} = b\frac{\partial u_3}{\partial x} \tag{3}$$

The solutions u_1 and u_3, which may be obtained from the author's equations (11), cannot satisfy the discussers' equation (3). Thus the author's claim appears to be invalid.

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[1] By Tien-Yu Tsui, published in the December, 1968, issue of the Journal of Applied Mechanics, Vol. 35, Trans. ASME, Vol. 90, Series E, pp. 824–825.

[2] Professor, School of Aerospace and Mechanical Engineering, University of Oklahoma, Norman, Okla. Mem. ASME.

[3] Assistant Professor, School of Aerospace and Mechanical Engineering, University of Oklahoma, Norman, Okla. Assoc. Mem. ASME.