the convective transport of heat is neglected in comparison to the conductive transport. Further, without obtaining the detailed steady-state base flow solution, the writers analyzed the instability in the fluid layer considering the base flow motion to be of negligible order. How far this is valid can be ascertained only by experimentation or further detailed analysis. The writers believe that the results obtained would be found correct within agreeable limits at least for small values of γ. For $\gamma = \infty$, i.e., for the horizontally heated case, departures from the values given in the article may be found as a result of the existence of the base flow.

There is no particular difficulty in including the base flow velocity in the stability analysis. However, to obtain the base flow pattern for such a system with aspect ratios, $d/l \ll 1$ (where d is the distance in the vertical direction and l distance in the horizontal direction) is very difficult. Elder[3] has previously obtained through experimentation the base flow pattern for a horizontally heated system with aspect ratios, $d/l \gg 1$. Furthermore, using finite-difference approximation to the governing differential equations, Elder[4] obtained theoretical results which verified his experimental values. In Elder[3] certain difficulties were mentioned in satisfying exactly the boundary conditions at the horizontal surfaces. The writers are at present engaged in developing a finite-element procedure to determine the base flow in a horizontally heated system with very small aspect ratios. Once this has been successfully carried out, the writers intend to include the base flow in the stability analysis for such systems.

[3] Elder, J. W., "Laminar Free Flow in a Vertical Slot," *Journal of Fluid Mechanics*, Vol. 23, Part 1, 1965, pp. 77–98.

[4] Elder, J. W., "Numerical Experiments With Free Convection in a Vertical Slot," *Journal of Fluid Mechanics*, Vol. 24, Part 4, 1966, pp. 823–843.

A Numerical Solution for the Laminar Wake Behind a Finite Flat Plate[1]

R. J. EL-ASSAR.[2] In their paper,[1] the authors present "an improved first approximation of the solution in the trailing-edge region for large values of the Reynolds number." It is surprising to note that the results in the authors' Figs. 7 and 8 show considerable sensitivity to the mesh size Δx. This is an indication that their numerical solution did not converge. The reported disagreement with Goldstein's results [1][3] in the immediate neighborhood of the trailing edge is, therefore, not substantiated. Indeed, the trend of their results in Fig. 7 shows that they might have achieved good agreement if they had employed a finer mesh. While the authors report that the size of the circular region around the trailing edge, where boundary-layer approximations are not valid, is estimated as $LR^{-3/4}$ which is extremely small for large Reynolds numbers, and while their major effort was to study the structure of the flow field within this region, they use a mesh size that is hundreds of times larger than this size. The trailing-edge disturbance is negligible outside the circular region of size $LR^{-3/4}$ and the use of the Navier-Stokes equations outside this region, although legitimate, only makes matters worse. This is due to the Drichlet-type boundary conditions necessary for the existence of the solution of the elliptic equations not being known a priori. One tends to believe that the boundary-layer equations are accurate enough to describe the whole flow field because of the rapid decay of the trailing-edge disturbance. The region where the influence of the trailing edge is significant is smaller than the mesh size in any efficient and stable numerical scheme [2]. We believe that the preasymptotic solution of Goldstein is valid and the choice between the Navier-Stokes equations and the boundary-layer equations depends on how close to the trailing edge one wants to extend the solution. Because of the parabolic nature of the boundary-layer equations any disturbance in the Blasius profile at the trailing edge (initial condition) due to the trailing-edge singularity would decay rapidly with no significant effect on the behavior of the near and far wakes. If one wants to study the behavior of the flow field in the immediate vicinity of the trailing-edge singularity, the method of series truncation as proposed by Van Dyke [3] should be used because of the rapid decay of the disturbance and, hence, the difficulty in achieving convergence in a difference scheme.

[1] By A. Plotkin and I. Flügge-Lotz, published in the December, 1968, issue of the Journal of Applied Mechanics, Vol. 35, Trans. ASME, Vol. 90, Series E, pp. 625–630.

[2] Assistant Research Professor, Department of Mechanical Engineering, Rutgers University, New Brunswick, N. J.

[3] Numbers in brackets designate References at end of Discussion.

References

1 Goldstein, S., "Concerning Some Solutions of the Boundary-Layer Equations in Hydrodynamics," *Proceedings of the Cambridge Philosophical Society*, Vol. 26, 1930, pp. 1–30.

2 El-Assar, R., and Page, R. H., "A Finite-Difference Solution of the Laminar Wake Behind a Flat Plate," paper presented at the Annual Meeting of the Division of Fluid Mechanics of APS, Seattle, Wash., Nov. 1968.

3 Van Dyke, M., "A Survey of Higher-Order Boundary-Layer Theory," SUDAER No. 326, Department of Aeronautics and Astronautics, Stanford University, Stanford, Calif., 1967.

Authors' Closure

It is not surprising that our results show some sensitivity to the mesh width Δx since decreasing the mesh width forces the necessity of making calculations closer to the trailing-edge singularity. This in itself does not indicate lack of convergence. In fact, the axial velocity component on the wake center line (Fig. 7) appears to be converging quite well, more rapidly even than the comparable convergent results for the boundary-layer solution as shown in Fig. 3.5 of Plotkin and Flügge-Lotz.[4] The more sensitive local shear stress coefficient (Fig. 8) appears to be converging (albeit slowly) in the region closest to the trailing edge. A close look at Fig. 7 will show that the trend of convergence is not toward the Goldstein[5] result closest to the trailing edge, as is stated by El-Assar.

The size of the region in which the boundary-layer approximations are not valid was estimated to be $LR^{-3/4}$ by Van Dyke.[6] At the time of publication, this estimate was as yet not substantiated by further analysis and to the authors' knowledge has still not been. The authors were therefore justified in searching for the trailing-edge disturbance in a larger region. We note that the size of the disturbance region as calculated in our paper is independent of mesh width for a given Reynolds number.

El-Assar "tends to believe that the boundary-layer equations are accurate enough to describe the whole flow field because of the rapid decay of the trailing-edge disturbance." Since the interest in this problem is primarily of an academic nature, the authors take exception to this statement. The initial conditions for the wake problem are only determined from a knowledge of the flow in the trailing-edge region, and without this knowledge can only be approximated. The numerical solution in the neighborhood of the assumed initial profile is therefore uncertain and, although the Goldstein solution may be approached downstream, it is not

[4] Plotkin, A., and Flügge-Lotz, I., "A Numerical Solution for the Laminar Wake Behind a Finite Flat Plate," Tech. Rep. No. 179, Division of Engineering Mechanics, Stanford University, Stanford, Calif., 1968.

[5] Goldstein, S., "Concerning Some Solutions of the Boundary Layer Equations in Hydrodynamics," *Proceedings of the Cambridge Philosophical Society*, Vol. 26, 1930, pp. 1–30.

[6] Van Dyke, M., "A Survey of Higher-Order Boundary-Layer Theory," SUDAER No. 326, Department of Aeronautics and Astronautics, Stanford University, Stanford, Calif., 1967.

known at what point the solution becomes valid. The authors feel that to correctly describe the flow field in the wake, a method of solution treating the complete flow region must be used. We attempted to do this numerically and appear to have had some success.

In closing, we would like to point out two papers which study the flow in the trailing-edge region in some detail. Talke and Berger[7] use the method of series truncation as suggested by Van Dyke, and Stewartson[8] uses an improved modified Oseen-type analysis.

[7] Talke, F., and Berger, S. A., "Analysis of the Laminar Incompressible Flow Near the Trailing Edge of a Flat Plate Using Series Truncation," paper presented at the annual meeting of the Division of Fluid Mechanics of APS, Seattle, Wash., Nov. 1968.

[8] Stewartson, K., "On the Flow Near the Trailing Edge of a Flat Plate," *Proceedings of the Royal Society*, Series A, Vol. 306, 1968, pp. 275–290.

The Strongest Circular Arch—A Perturbation Solution[1]

ROBERT SCHMIDT.[2] Professor Wu's paper is a valuable contribution to the store of knowledge of structural optimization. The perturbation technique used seems to yield useful results with a comparatively modest expenditure of effort. As mentioned by the author, a circular arch with clamped ends, which is subjected to a uniform external hydrostatic pressure, could also have been treated by the procedure described in the paper, though with greater difficulties. In the writer's opinion, hinged as well as clamped symmetrical arches under nonuniform loads could also be treated by the author's procedure, even though the classical theory of buckling of arches seems to be inapplicable for that kind of load because of large prebuckling deflections. However, for a practical range of values of the central angle 2α, the classical theory seems to predict buckling loads that are in fair agreement with exact results. Professor DaDeppo and the writer[3] have examined a very unfavorable case of load; namely, a concentrated load P at the crown of hinged-hinged circular arches. A semicircular arch under such a load buckles by sidesway at the time when the vertical deflection of the crown amounts to about 20 percent of the radius a. Certainly this is a large deflection. The exact value of the critical load, as calculated by means of elliptic integrals using the inextensional theory of circular elastica, is $5.8605\ EI/a^2$, as compared to $6.0EI/a^2$ obtained by the classical small-deflection theory of stability. Fair agreement also exists for arches that are shallower than the semicircular arch:

α	Exact Pa^2/EI	Classic Pa^2/EI
30 deg	22.66	23.45
40 deg	17.15	17.17
50 deg	13.82	13.33
60 deg	11.44	10.69
70 deg	9.46	8.74
80 deg	7.62	7.22
90 deg	5.86	6.00
100 deg	4.29	4.98

The reason for this unexpected agreement is obscure. To consider it as mere coincidence is difficult, since the similarity of values extends over a wide range. Most likely the agreement is due to a high value of fairly constant prebuckling thrust.

As seen from the author's Fig. 1, the calculated depth of the "strongest" arch rib vanishes at the two hinged supports and at the crown; i.e., at points at which the bending couples are zero. To transmit the thrust, the depth must not be zero. For that reason, it might have been better to have used the expression $h_c + h(x)$ to describe the depth of the arch rib, together with the extensional theory of arches.

The weight of the arch frequently is not an insignificant factor. In future researches, it should be included in the equations. In view of the foregoing discussion, inclusion of the weight might not invalidate the usefulness of the classical eigenvalue theory of stability for certain values of the central angle.

[1] By C.-H. Wu, published in the September, 1968, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 35, TRANS. ASME, Vol. 90, Series E, pp. 476–480.

[2] Professor of Engineering Mechanics, University of Detroit, Detroit, Mich.

[3] Schmidt, R., and DaDeppo, D. A., "On Buckling of Deep Arches at Large Deflections," submitted for publication in the *AIAA Journal*.

Author's Closure

The author thanks Professor Schmidt for his kind comments.

The method of perturbation is a powerful tool for the determination of the solution of a problem containing a small parameter ϵ for which the reduced problem ($\epsilon = 0$) exists and has a simple solution. The reduced problem for the case of a circular arch under uniform pressure is a fictitious column under axial load. It is not clear to the author that such a reduced problem exists for the case of a circular arch under arbitrary load.

Wave Propagation in a Finite-Length Bar With a Variable Cross Section[1]

C. W. BERT[2] **and D. M. EGLE.**[3] The writers object to the author's claim that the solution of the author's equation (2) for an arbitrary variation in cross section of the form

$$A = A_0 \sum_{n=1}^{\infty} a_n (x/\mu l)^n$$

may be obtained by superimposing the solutions obtained for each term in the series. The fallacy of this claim may be shown by considering the following two-term series as an example:

$$A = A_0(x + x^3)$$

If u_1 and u_3 are the solutions of the author's equation (3) with $n = 1, 3$, respectively, then the author's claim is that

$$u = u_1 + bu_3 \tag{1}$$

where b is a constant, is a solution of

$$\frac{\partial}{\partial x}\left[(x + x^3)\frac{\partial u}{\partial x}\right] = \frac{1}{c^2}(x + x^3)\frac{\partial^2 u}{\partial t^2} \tag{2}$$

Substitution of (1) into (2) will show that this can be true only if

$$x^2 \frac{\partial u_1}{\partial x} = b\,\frac{\partial u_3}{\partial x} \tag{3}$$

The solutions u_1 and u_3, which may be obtained from the author's equations (11), cannot satisfy the discussers' equation (3). Thus the author's claim appears to be invalid.

[1] By Tien-Yu Tsui, published in the December, 1968, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 35, TRANS. ASME, Vol. 90, Series E, pp. 824–825.

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