$$C_i^* \equiv \frac{\partial C_i}{\partial R_i} = \frac{1}{\pi}\left[\mu_i\theta_i^* + (1 - \mu_i)\pi + \frac{\mu_i}{2}\sin 2\theta_i^* - 2\mu_i \sin\theta_i^*\right]$$

$$S_i^* \equiv \frac{\partial S_i}{\partial R_i} = -\frac{4\mu_i}{\pi(R_i^0)^2} \qquad (i = 1, 2)$$

It is found that the portion of the steady-state amplitude and frequency curves with negative slopes give at least one eigenvalue with positive real parts. The portions of the steady-state oscillations curves with negative slopes are in fact unstable branches. Therefore, there is only *one* state of stable steady-state oscillations under a load less than the critical load.

Author's Closure

The author extends his thanks to Professor Tso and Mr. Asmis for their comments.

It is true that unless the angular rotation exceeds the value a, the system analyzed by the author is identical to an undamped system; however, it should be noted that the parameter a used in the investigation can be assigned any small value greater than zero. In other words, if the initial disturbances and ϕ_i are of the first-order infinitesimal quantities, we may assign a a magnitude which is greater than zero but is smaller than the first-order infinitesimal quantities so that the hysteretic effects will still be present in the system. Since this reasoning is valid for any small disturbances and ϕ_i, the notion of "stability in the large" used in the hysteretic damping case is rather weak.

To throw additional light on problems concerning the notion of stability needed in the hysteretic damping case, let us consider the case in which a distributed-yielding hysteretic model is used in the representation of the restoring and dissipative mechanisms in the hinges of the double pendulum. The hysteretic restoring moments induced may, then, be written as functionals $cM_i[\phi_i, \mu_i, t, Y_i(a)]$ where $i = 1, 2$, and the functions $Y_i(a)$ define, respectively, the fraction (or percentage) of the material elements with yield limit equal to a in the hinge. Assume that

$$Y_i(a) = \frac{1}{a^*}\,[\mathrm{H}(a) - \mathrm{H}(a - a^*)] \qquad (i = 1, 2)$$

where a^* is the highest yield limit of the material elements in the hinge and H( ) is the Heavyside's unit step function. The total area of the yield-limit distribution diagram so specified is unity. The relationship between the restoring moment and the angular displacement may implicitly be depicted by Fig. 2 of the paper with the labeling of the ordinate $M_i(\phi_i, \mu_i, t)$ replaced by $\frac{1}{Y_i(a)}\frac{d}{da}M_i[\phi_i, \mu_i, t, Y_i(a)]$. It can be shown that the resultant system hysteresis loops are composed of smooth curves whose curvatures are equal to $\frac{1}{2}\mu_i Y_i((A_i - \phi_i)/2)$ for the lower branch curves, and $-\frac{1}{2}\mu_i Y_i((A_i + \phi_i)/2)$ for the upper branch curves. (See ref. [18] in the paper and [1] given in this Closure.)

The distributed-yielding hysteretic model just described will, no doubt, exhibit hysteretic damping effects whenever the system oscillates with any small amplitudes. This illustration shows that the notion of "stability in the small" is applicable in the hysteretic damping case as well as in the viscous damping case.

In a recent study ([2] of this Closure), the distributed-yielding hysteretic damping was found to have similar destabilizing effects as the bilinear hysteretic damping. Thus the hysteretic damping may exert destabilizing effects on stability of the system "in the small."

With respect to the latter part of the discussion, the author would like to express his appreciation to the discussers for bringing out the stability analysis of the steady-state curves which was not emphasized in the paper. However, it should be noted that the author has indicated only the possibility that two disparate states of steady-state oscillations of the system may exist under a certain identical loading. There was no assertion that both of these two states of steady-state oscillations are stable.

References

1 Iwan, W. D., "On a Class of Models for the Yielding Behavior of Continuous and Composite Systems," Journal of Applied Mechanics, Vol. 34, No. 3, Trans. ASME, Vol. 89, Series E, Sept. 1967, pp. 612–617.

2 Jong, I. C., "On Stability of a Circulatory System With Weak Distributed Yielding Effects," to be published as a Brief Note in the Journal of Applied Mechanics.

Thermal Instability in Fluid Layers in the Presence of Horizontal and Vertical Temperature Gradients[1]

S. H. DAVIS.[2] The authors wish to consider the effects of both horizontal and vertical temperature gradients on the motion of fluid in a two-dimensional slot. They use the methods of linear hydrodynamic stability to analyze the basic state of zero velocity with the temperature profile

$$T = T_0 - \beta_x \cdot x - \beta_z \cdot z. \tag{1}$$

Perturbations are superposed on this state and critical Rayleigh numbers (for the onset of motion) are computed.

The basic state (1), however, is not a solution of the basic hydrodynamic equations. The proper basic equilibrium state requires zero velocity and the temperature and pressure (p) fields must satisfy

$$\nabla^2 T = 0 \tag{2}$$

$$\nabla p + g\rho\bar{k} = 0 \tag{3}$$

where we have the equation of state

$$\rho = \rho_0[1 - \alpha(T - T_0)]. \tag{4}$$

Here $\bar{k} = (0, 1)$, g is the acceleration of gravity, α is the volume expansion coefficient, and the zero subscript denotes a (constant) reference value of the corresponding quantity.

The temperature field (1) does satisfy equation (2) but not equation (3). To see this, take the curl of equation (3) and obtain

$$0 = \nabla \times \rho\bar{k} = \nabla\rho \times \bar{k}. \tag{5}$$

Equation (5) can only be satisfied if ρ is independent of x. From equation (4), this implies that T is independent of x.

Since basic equilibrium states exist only when $T = T(z)$ only, the stability of state (1) cannot be discussed. In fact, *any* horizontal temperature gradient should be sufficient to induce motion.

Authors' Closure

The writers appreciate the discussion by Mr. Davis and agree with the points raised by him which the writers consider to be valid for the most general case. However, one of the aims of the writers was to demonstrate the effect of small horizontal temperature gradients on the instability in adversely heated fluid layers of high thermal conductivity. Thus, in the energy equation,

[1] By T. E. Unny and P. Niessen, published in the March, 1969, issue of the Journal of Applied Mechanics, Vol. 36, Trans. ASME, Vol. 91, Series E, pp. 121–123.

[2] Assistant Professor of Mechanics, The Johns Hopkins University, Baltimore, Md.

the convective transport of heat is neglected in comparison to the conductive transport. Further, without obtaining the detailed steady-state base flow solution, the writers analyzed the instability in the fluid layer considering the base flow motion to be of negligible order. How far this is valid can be ascertained only by experimentation or further detailed analysis. The writers believe that the results obtained would be found correct within agreeable limits at least for small values of γ. For $\gamma = \infty$, i.e., *for the horizontally heated case, departures from the values given* in the article may be found as a result of the existence of the base flow.

There is no particular difficulty in including the base flow velocity in the stability analysis. However, to obtain the base flow pattern for such a system with aspect ratios, $d/l \ll 1$ (where d is the distance in the vertical direction and l distance in the horizontal direction) is very difficult. Elder[3] has previously obtained through experimentation the base flow pattern for a horizontally heated system with aspect ratios, $d/l \gg 1$. Furthermore, using finite-difference approximation to the governing differential equations, Elder[4] obtained theoretical results which verified his experimental values. In Elder[3] certain difficulties were mentioned in satisfying exactly the boundary conditions at the horizontal surfaces. The writers are at present engaged in developing a finite-element procedure to determine the base flow in a *horizontally heated system with very small aspect ratios.* Once this has been successfully carried out, the writers intend to include the base flow in the stability analysis for such systems.

[3] Elder, J. W., "Laminar Free Flow in a Vertical Slot," *Journal of Fluid Mechanics*, Vol. 23, Part 1, 1965, pp. 77–98.

[4] Elder, J. W., "Numerical Experiments With Free Convection in a Vertical Slot," *Journal of Fluid Mechanics*, Vol. 24, Part 4, 1966, pp. 823–843.

A Numerical Solution for the Laminar Wake Behind a Finite Flat Plate[1]

R. J. EL-ASSAR.[2] In their paper,[1] the authors present "an improved first approximation of the solution in the trailing-edge region for large values of the Reynolds number." It is surprising to note that the results in the authors' Figs. 7 and 8 show considerable sensitivity to the mesh size Δx. This is an indication that their numerical solution did not converge. The reported disagreement with Goldstein's results [1][3] in the immediate neighborhood of the trailing edge is, therefore, not substantiated. Indeed, the trend of their results in Fig. 7 shows that *they might have achieved good agreement if they had employed* a finer mesh. While the authors report that the size of the circular region around the trailing edge, where boundary-layer approximations are not valid, is estimated as $LR^{-3/4}$ which is extremely small for large Reynolds numbers, and while their major effort was to study the structure of the flow field within this region, they use a mesh size that is hundreds of times larger than this size. The trailing-edge disturbance is negligible outside the circular region of size $LR^{-3/4}$ and the use of the Navier-Stokes equations outside this region, although legitimate, only makes matters worse. This is due to the Drichlet-type boundary conditions necessary for the existence of the solution of the elliptic equations not being known a priori. One tends to believe that the boundary-layer equations are accurate enough to describe the whole flow field because of the rapid decay of the trailing-edge disturbance. The region where the influence of the trailing edge is significant is smaller than the mesh size in any efficient and stable numerical scheme [2]. We believe that the preasymptotic solution of Goldstein is valid and the choice between the Navier-Stokes equations and the boundary-layer equations depends on how close to the trailing edge one wants to extend the solution. Because of the parabolic nature of the boundary-layer equations any disturbance in the Blasius profile at the trailing *edge (initial condition) due to the trailing-edge singularity would* decay rapidly with no significant effect on the behavior of the near and far wakes. If one wants to study the behavior of the flow field in the immediate vicinity of the trailing-edge singularity, the method of series truncation as proposed by Van Dyke [3] should be used because of the rapid decay of the disturbance and, hence, the difficulty in achieving convergence in a difference scheme.

[1] By A. Plotkin and I. Flügge-Lotz, published in the December, 1968, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 35, TRANS. ASME, Vol. 90, Series E, pp. 625–630.

[2] Assistant Research Professor, Department of Mechanical Engineering, Rutgers University, New Brunswick, N. J.

[3] Numbers in brackets designate References at end of Discussion.

References

1 Goldstein, S., "Concerning Some Solutions of the Boundary-Layer Equations in Hydrodynamics," *Proceedings of the Cambridge Philosophical Society*, Vol. 26, 1930, pp. 1–30.

2 El-Assar, R., and Page, R. H., "A Finite-Difference Solution of the Laminar Wake Behind a Flat Plate," paper presented at the Annual Meeting of the Division of Fluid Mechanics of APS, Seattle, Wash., Nov. 1968.

3 Van Dyke, M., "A Survey of Higher-Order Boundary-Layer Theory," SUDAER No. 326, Department of Aeronautics and Astronautics, Stanford University, Stanford, Calif., 1967.

Authors' Closure

It is not surprising that our results show some sensitivity to the mesh width Δx since decreasing the mesh width forces the necessity of making calculations closer to the trailing-edge singularity. This in itself does not indicate lack of convergence. In fact, the axial velocity component on the wake center line (Fig. 7) appears to be converging quite well, more rapidly even than the comparable convergent results for the boundary-layer solution as shown in Fig. 3.5 of Plotkin and Flügge-Lotz.[4] The more sensitive local shear stress coefficient (Fig. 8) appears to be converging (albeit slowly) in the region closest to the trailing edge. A close look at Fig. 7 will show that the trend of convergence is not toward the Goldstein[5] result closest to the trailing edge, as is stated by El-Assar.

The size of the region in which the boundary-layer approximations are not valid was estimated to be $LR^{-3/4}$ by Van Dyke.[6] At the time of publication, this estimate was as yet not substantiated by further analysis and to the authors' knowledge has still not been. The authors were therefore justified in searching for the trailing-edge disturbance in a larger region. We note that the size of the disturbance region as calculated in our paper is independent of mesh width for a given Reynolds number.

El-Assar "tends to believe that the boundary-layer equations are accurate enough to describe the whole flow field because of the rapid decay of the trailing-edge disturbance." Since the interest in this problem is primarily of an academic nature, the authors take exception to this statement. The initial conditions for the wake problem are only determined from a knowledge of the flow in the trailing-edge region, and without this knowledge can only be approximated. The numerical solution in the neighborhood of the assumed initial profile is therefore uncertain and, although the Goldstein solution may be approached downstream, it is not

[4] Plotkin, A., and Flügge-Lotz, I., "A Numerical Solution for the Laminar Wake Behind a Finite Flat Plate," Tech. Rep. No. 179, Division of Engineering Mechanics, Stanford University, Stanford, Calif., 1968.

[5] Goldstein, S., "Concerning Some Solutions of the Boundary Layer Equations in Hydrodynamics," *Proceedings of the Cambridge Philosophical Society*, Vol. 26, 1930, pp. 1–30.

[6] Van Dyke, M., "A Survey of Higher-Order Boundary-Layer Theory," SUDAER No. 326, Department of Aeronautics and Astronautics, Stanford University, Stanford, Calif., 1967.