discussion

On Stability of a Circulatory System With Bilinear Hysteresis Damping[1]

W. K. TSO[2] **and K. G. ASMIS.**[3] Professor Jong has considered a very interesting problem. The purpose of this discussion is to clarify two points in his paper.

The author compares the effects of viscous damping and hysteretic damping on the critical load of a nonconservative system. It should be pointed out that in the case of viscous damping, the notion of stability used is "stability in the small." Whether the system is stable or not depends on the subsequent motion of the system after it is subjected to infinitesimal disturbances. On the other hand, the notion of stability used in the hysteretic damping case is "stability in the large." Unless the disturbances (finite) are sufficiently large so that the angular rotations exceed the value "a," the system analyzed by the author is identical to an undamped system.

For example, consider the case where the hysteretic damping values correspond to $\delta_2 = 6$ deg and $\xi = 5$. From the author's Fig. 4, one can interpret that when $F = 1.75$, the system is stable both in the small and in the large. When $F = 1.85$, the author indicates that the system is unstable due to the destabilizing of hysteretic damping. It is true that the system is unstable in the large, but the system is *stable* in the small since the load F is less than $F_c = 2.086$. Only then when $F > F_c$ is the system unstable, both in the small and in the large. Since the viscous damping has destabilizing effect on stability of the system subjected to infinitesimal disturbances, while hysteretic damping has destabilizing effect on stability of the system in the large, proper interpretation of the result is necessary when comparing the destabilizing effects of viscous damping and hysteretic damping.

The second point concerns the stability of the steady-state curves as given in the author's Figs. 4 and 5. The author indicates that two disparate states of steady-state oscillations under the same loading exist. A stability analysis of the steady-state curves shows that the portions of the curve with negative slope in Figs. 4 and 5 are in fact unstable branches. The stability analysis of the steady-state curves is given as follows. Let

$$\bar{R}_1 = R_1^0 + \zeta_1 \tag{1}$$

$$\bar{R}_2 = R_2^0 + \zeta_2 \tag{2}$$

$$\bar{\psi}_1 = \psi_1^0 + \eta_1 \tag{3}$$

$$\bar{\psi}_2 = \psi_2^0 + \eta_2 \tag{4}$$

where R_i^0 and ψ_i^0 $(i = 1, 2)$ are the steady-state amplitudes and phase angles. ζ_i and η_i $(i = 1, 2)$ are infinitesimal perturbations of the amplitudes and phase angles from the steady-state values.

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[1] By I. C. Jong, published in the March, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 76–82.

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Substituting equations (1)–(4) in the author's equations (23) and (24), noting that R_i^0 and ψ_i^0 $(i = 1, 2)$ satisfy equation (32) and (33) of the author, and after neglecting higher-order terms in ζ_i and η_i $(i = 1, 2)$, the following result is obtained

$$[D] \begin{Bmatrix} \zeta_1' \\ \eta_1' \\ \zeta_2' \\ \eta_2' \end{Bmatrix} = [B] \begin{Bmatrix} \zeta_1 \\ \eta_1 \\ \zeta_2 \\ \eta_2 \end{Bmatrix} \tag{5}$$

where $[D]$ is a diagonal matrix with elements $D_{11} = 4\omega$, $D_{22} = 4\omega R_1^0$, $D_{33} = -2\omega$, and $D_{44} = 2\omega R_2^0$.

The elements in matrix B are listed as follows:

$$B_{11} = S_1^*$$
$$B_{12} = (2C_2^0 - FR_2^0) \cos \Psi + 2S_2^0 \sin \Psi$$
$$B_{13} = (2C_2^* - F) \sin \Psi - 2S_2^* \cos \Psi$$
$$B_{14} = (FR_2^0 - 2C_2^0 \cos \Psi) - 2S_2^0 \sin \Psi$$
$$B_{21} = 2\omega^2 - C_1^*$$
$$B_{22} = 2S_2^0 \cos \Psi + (FR_2^0 - 2C_2^0) \sin \Psi$$
$$B_{23} = 2S_2^* \sin \Psi + (2C_2^* - F) \cos \Psi$$
$$B_{24} = -2S_2^0 \cos \Psi + (2C_2^0 - FR_2^0) \sin \Psi$$
$$B_{31} = C_1^* \sin \Psi + S_1^* \cos \Psi$$
$$B_{32} = C_1^0 \cos \Psi - S_1^0 \sin \Psi$$
$$B_{33} = -3S_2^*$$
$$B_{34} = S_1^0 \sin \Psi - C_1^0 \cos \Psi$$
$$B_{41} = C_1^* \cos \Psi - S_1^* \sin \Psi$$
$$B_{42} = -S_1^0 \cos \Psi - C_1^0 \sin \Psi$$
$$B_{43} = \omega^2 - 3C_2^* + F$$
$$B_{44} = S_1^0 \cos \Psi + C_1^0 \sin \Psi$$

where

$$S_i^* \equiv \frac{\partial S_i}{\partial R_i} \quad (i = 1, 2)$$

$$C_i^* \equiv \frac{\partial C_i}{\partial R_i} \quad (i = 1, 2)$$

$$\Psi \equiv \psi_1^0 - \psi_2^0$$

It is implied in equation (5) that all quantities in the matrices D and B are to be evaluated at the steady-state values. Equation (5) is a system of first-order linear differential equations. The stability of the solution is governed by the eigenvalues of the matrix $D^{-1}B$. The solution of (5), and hence the steady-state amplitude and frequency, is stable if, and only if, the eigenvalues are either simple with nonpositive real parts or have negative real parts.

To determine the stability of the steady-state oscillations, the values of F, ω, R_1^0, and R_2^0 were scaled from the author's Figs. 3, 4, and 5. C_i^0, S_i^0, and θ_i^* $(i = 1, 2)$ were calculated from the author's equations (38), (39), and (40). Sin Ψ and cos Ψ were obtained from the author's equation (33). The derivatives C_i^* and S_i^* are given by reference [13] of the author's paper.

$$C_i{}^* \equiv \frac{\partial C_i}{\partial R_i} = \frac{1}{\pi}\left[\mu_i\theta_i{}^* + (1 - \mu_i)\pi\right.$$

$$\left. + \frac{\mu_i}{2}\sin 2\theta_i{}^* - 2\mu_i\sin\theta_i{}^*\right]$$

$$S_i{}^* \equiv \frac{\partial S_i}{\partial R_i} = -\frac{4\mu_i}{\pi(R_i{}^0)^2} \qquad (i = 1, 2)$$

It is found that the portion of the steady-state amplitude and frequency curves with negative slopes give at least one eigenvalue with positive real parts. The portions of the steady-state oscillations curves with negative slopes are in fact unstable branches. Therefore, there is only one state of stable steady-state oscillations under a load less than the critical load.

Author's Closure

The author extends his thanks to Professor Tso and Mr. Asmis for their comments.

It is true that unless the angular rotation exceeds the value a, the system analyzed by the author is identical to an undamped system; however, it should be noted that the parameter a used in the investigation can be assigned any small value greater than zero. In other words, if the initial disturbances and ϕ_i are of the first-order infinitesimal quantities, we may assign a a magnitude which is greater than zero but is smaller than the first-order infinitesimal quantities so that the hysteretic effects will still be present in the system. Since this reasoning is valid for any small disturbances and ϕ_i, the notion of "stability in the large" used in the hysteretic damping case is rather weak.

To throw additional light on problems concerning the notion of stability needed in the hysteretic damping case, let us consider the case in which a distributed-yielding hysteretic model is used in the representation of the restoring and dissipative mechanisms in the hinges of the double pendulum. The hysteretic restoring moments induced may, then, be written as functionals $cM_i[\phi_i, \mu_i, t, Y_i(a)]$ where $i = 1, 2$, and the functions $Y_i(a)$ define, respectively, the fraction (or percentage) of the material elements with yield limit equal to a in the hinge. Assume that

$$Y_i(a) = \frac{1}{a^*}\left[\mathrm{H}(a) - \mathrm{H}(a - a^*)\right] \qquad (i = 1, 2)$$

where a^* is the highest yield limit of the material elements in the hinge and H( ) is the Heavyside's unit step function. The total area of the yield-limit distribution diagram so specified is unity. The relationship between the restoring moment and the angular displacement may implicitly be depicted by Fig. 2 of the paper with the labeling of the ordinate $M_i(\phi_i, \mu_i, t)$ replaced by $\frac{1}{Y_i(a)}\frac{d}{da}M_i[\phi_i, \mu_i, t, Y_i(a)]$. It can be shown that the resultant system hysteresis loops are composed of smooth curves whose curvatures are equal to $\frac{1}{2}\mu_iY_i((A_i - \phi_i)/2)$ for the lower branch curves, and $-\frac{1}{2}\mu_iY_i((A_i + \phi_i)/2)$ for the upper branch curves. (See ref. [18] in the paper and [1] given in this Closure.)

The distributed-yielding hysteretic model just described will, no doubt, exhibit hysteretic damping effects whenever the system oscillates with any small amplitudes. This illustration shows that the notion of "stability in the small" is applicable in the hysteretic damping case as well as in the viscous damping case.

In a recent study ([2] of this Closure), the distributed-yielding hysteretic damping was found to have similar destabilizing effects as the bilinear hysteretic damping. Thus the hysteretic damping may exert destabilizing effects on stability of the system "in the small."

With respect to the latter part of the discussion, the author would like to express his appreciation to the discussers for bringing out the stability analysis of the steady-state curves which was not emphasized in the paper. However, it should be noted that

the author has indicated only the possibility that two disparate states of steady-state oscillations of the system may exist under a certain identical loading. There was no assertion that both of these two states of steady-state oscillations are stable.

References

1 Iwan, W. D., "On a Class of Models for the Yielding Behavior of Continuous and Composite Systems," JOURNAL OF APPLIED MECHANICS, Vol. 34, No. 3, TRANS. ASME, Vol. 89, Series E, Sept. 1967, pp. 612–617.

2 Jong, I. C., "On Stability of a Circulatory System With Weak Distributed Yielding Effects," to be published as a Brief Note in the JOURNAL OF APPLIED MECHANICS.

Thermal Instability in Fluid Layers in the Presence of Horizontal and Vertical Temperature Gradients[1]

S. H. DAVIS.[2] THE authors wish to consider the effects of both horizontal and vertical temperature gradients on the motion of fluid in a two-dimensional slot. They use the methods of linear hydrodynamic stability to analyze the basic state of zero velocity with the temperature profile

$$T = T_0 - \beta_x\cdot x - \beta_z\cdot z. \qquad (1)$$

Perturbations are superposed on this state and critical Rayleigh numbers (for the onset of motion) are computed.

The basic state (1), however, is not a solution of the basic hydrodynamic equations. The proper basic equilibrium state requires zero velocity and the temperature and pressure (p) fields must satisfy

$$\nabla^2 T = 0 \qquad (2)$$

$$\nabla p + g\rho\bar{k} = 0 \qquad (3)$$

where we have the equation of state

$$\rho = \rho_0[1 - \alpha(T - T_0)]. \qquad (4)$$

Here $\bar{k} = (0, 1)$, g is the acceleration of gravity, α is the volume expansion coefficient, and the zero subscript denotes a (constant) reference value of the corresponding quantity.

The temperature field (1) does satisfy equation (2) but not equation (3). To see this, take the curl of equation (3) and obtain

$$0 = \nabla \times \rho\bar{k} = \nabla\rho \times \bar{k}. \qquad (5$$

Equation (5) can only be satisfied if ρ is independent of x. From equation (4), this implies that T is independent of x.

Since basic equilibrium states exist only when $T = T(z)$ only, the stability of state (1) cannot be discussed. In fact, any horizontal temperature gradient should be sufficient to induce motion.

Authors' Closure

The writers appreciate the discussion by Mr. Davis and agree with the points raised by him which the writers consider to be valid for the most general case. However, one of the aims of the writers was to demonstrate the effect of small horizontal temperature gradients on the instability in adversely heated fluid layers of high thermal conductivity. Thus, in the energy equation,