### Author's Closure

The author wishes to thank Dr. Dengler for his interest in the paper.

Dr. Dengler's remark, that the difficulties of establishing boundary conditions in the "total deflection approach" are over-emphasized, comes as somewhat of a surprise to the author. In Dr. Dengler's own work the difficulties of defining proper boundary conditions at a beam cut, for the Timoshenko theory, was a subject of great concern.[4] In that work it is stated that the indefinite character of the boundary conditions was the primary reason for treating the infinite-beam problem by a method that did not require their definition. With this, and the Uflyand boundary condition errors reference (2), as examples, one can hardly refer to these difficulties as overemphasized.

Further, the author is not convinced that the suggested alternate approach, for the problem originally treated in reference (3), would lead to a correct solution. It is highly doubtful that one could justify the use of the stated conditions at $x = 0$. From Equations [2] of the paper it is obvious that

$$y_x(x, t) = y_{bx}(x, t) + y_{sx}(x, t)$$

where $y_b$ and $y_s$ are the bending and shear deflections, respectively. $y_x(0, t) = 0$ requires that $y_{bx}(0, t) = 0$, and $y_{sx}(0, t) = 0$, since obviously the special case where $y_{bx}(0, t) = -y_{sx}(0, t)$ is of no concern here. The author would agree that $y_{bx}(0, t) = 0$, but not that $y_{sx}(0, t) = 0$. In the limit (as $\epsilon \to 0$) the action of $q(x, t)$ tends toward that of a concentrated force at $x = 0$. In view of the relation for the shear force, Equation [3b] of the paper

$$S(x, t) = -k'A_0 G y_{sx}(x, t)$$

$y_{sx}(0, t)$, then, would not be zero.

Loss should not be made of the fact that the method of the paper leads to the solution of the infinite-beam problem of reference (3) in a more direct way than methods that employ the nonhomogeneous Timoshenko equation. Only a single Laplace transformation is involved. The moment transform $\bar{m}(x, p)$ of reference (3)[5] can be obtained by letting $S(0, t)$, in Equations [13] of the paper, be the Dirac function $\delta(t)$ with an associated impulse of magnitude $A$.

The author regrets not having seen the Leonard and Budiansky work[3] in time to comment in the paper on its important part in this subject.

# Dynamic Stress-Strain Relations for Annealed 2S Aluminum Under Compression Impact[1]

**J. D. Campbell.[2]** The authors are to be congratulated on the interesting experimental work described in this paper, and it is gratifying to note the similarity between the authors' results and those obtained by the writer using a similar method of test.

The difference between the two dynamic stress-strain curves $A$ and $B$ in Fig. 4 of the paper appears to be not more than 10 per cent in strain at any stress. This corresponds to about 5 per cent in velocity in the curve of Fig. 2; i.e., to give agreement

the curve of Fig. 2 should be raised by about 5 per cent. It appears from the points given in Fig. 2 that the curve could well be drawn rather higher, particularly near its upper end. Therefore it would be interesting to know the authors' estimate of the accuracy with which the experimental curves in Figs. 2 and 3 can be established.

The authors state that the difference between their three postulated stress-strain curves, Fig. 8, results from different rates of strain. It seems to the writer, however, that the difference is too large to be attributed to this cause. The authors estimated the minimum loading time in their tests to be about 2 microsec, and thus the mean rate of strain for the curve marked "5 per cent final strain" (Fig. 8) is about 25,000 sec$^{-1}$. The particle velocity for the "2 per cent final strain" curve is about half that for the "5 per cent final strain" curve; thus, assuming that the rise time is inversely proportional to the particle velocity, the mean rate of strain may be estimated to be about 5000 sec$^{-1}$. The authors also give the stress-strain curve for a strain rate of 0.040 per min, i.e., 0.00067 sec$^{-1}$. The following table thus gives the stress at 2 per cent strain at the three strain rates:

| Strain rate, sec$^{-1}$ | 0.00067 | 5000 | 25000 |
|---|---|---|---|
| Stress at 2 per cent strain, psi | 8100 | 9500 | 10100 |

From this it will be seen that between the first and second columns the strain rate increases $7.5 \times 10^6$ times, and the stress about 17 per cent; the corresponding increases between the second and third columns are 5 times and 6 per cent. It seems unlikely that the second relatively small increase in log (rate of strain) could cause an increase of stress of the same order as that caused by the first very large increase.

Referring to the propagation distances of maximum strains quoted in Table 1 of the paper, it is difficult to see how the distance could be greater for a strain of 5 per cent than it is for a strain of 3.5 per cent; both sets of calculated distances show a reduction of distance as the strain increases. It appears, therefore, that accurate experimental determination of the distance is not possible; this is presumably due to the gradual fall of strain with distance shown in Fig. 7. The exact shape of the strain-distribution curve will depend on lateral inertia effects which are neglected in the theory.

**E. H. Lee.[3]** The authors are to be congratulated on the careful experimental measurements which provide new information concerning the dynamic response of metals. There still seem to be difficulties in interpreting the detail differences between the dynamic stress-strain curves associated with impacts with different maximum strain values. The interpretation given in the paper of having dynamic stress-strain relations which differ throughout the entire range of plastic strain is contradictory to the wave solution used, in which the front lower stress part of the wave is not influenced by the higher stress region following. This property is illustrated in Fig. 5 of the paper in which the front part of the wave applies for all three solutions.

As mentioned by the authors, the range of strain rate during a single impact is very wide. It is much wider than the average difference between impacts at the three strain values. This again suggests the difficulties involved in using three dynamic stress-strain curves. It must be borne in mind when considering the high stress-rate values at the impact surface estimated in the paper, that they are followed by a period during which high stress is maintained at almost constant magnitude. It may be that this period of maintained stress, which differs for each strain magnitude, has a more marked influence on the strain produced than the differences in initial loading rate.

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[4] See particularly the paragraph following Equation [22], reference (3) of the paper, p. 183.

[5] See Equation [39].

[1] By J. E. Johnson, D. S. Wood, and D. S. Clark, published in the December, 1953, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 75, pp. 523–529.

[2] Lecturer in Engineering Science, Oxford University, Oxford, England.

[3] Professor of Applied Mathematics, Graduate Division, Brown University, Providence, R. I. Mem. ASME

In spite of these difficulties in the present analysis, the writer feels that this paper advances our knowledge of this difficult question and hopes that the authors are undertaking further work to enable the problem of the detail of the strain-rate influence to be elucidated.

As a question of detail, it is not apparent from Fig. 8 that the elastic limit constant $\epsilon_0$ differs for the three curves. Some information on this variation would help in assessing these results.

L. E. MALVERN.[4]  The authors have made a valuable addition to the existing information about a very puzzling phenomenon, the nature of the time dependence or rate dependence of the stress-strain relation in impact. The stress-time record at the impact end shows essentially constant stress until the arrival of unloading waves reflected from the free end of the specimen. This seems to indicate that the departure of the maximum stress and strain from the values predicted by the von Kármán theory, using the static stress-strain relation, cannot be explained by postulating a relaxation type of time dependence.

Some additional static compression of specimens cut from the constant residual-strain region of the bars used in the impact test might give further useful information. If in the further compression, the stress-strain curve is appreciably higher than the continuation of the original static curve, a dependence of the hardening on strain-rate history would be indicated, possibly due to greater disordering of the crystalline structure in the deformation under impact. Stress relaxation would be expected if the dependence is on instantaneous strain rate, rather than strain-rate history. But the uniformity of the residual strain for a considerable distance from the impact end seems to defy explanation on the basis of history dependence, since the strain-rate history varies greatly with distance from the impact end.

It would be desirable to have strain-time records obtained at stations on the specimen. For impact velocities much lower than most of those used by the authors, Sternglass and Stuart[5] have obtained such records for tensile impacts on specimens under initial tension.

Their results appear to show that in the transient response to impact, small plastic-strain increments are propagated at the elastic-wave velocity even when the initial tension is well into the plastic range. These results do not appear to be explainable either by a single dynamic stress-strain curve or by a family of curves as proposed in the present paper.

D. A. STUART.[6]  The authors are to be congratulated for their fine work and carefully written paper. However, the writer has some questions and comments which may concern, or be of interest in, their work.

It seems unfortunate that the authors did not determine the strain as a function of time in the test specimen. Using SR-4 strain gages, this procedure is not difficult and is extremely valuable. For example, using such a technique in the investigation of transient pulses in annealed copper, Sternglass and Stuart[5] have observed recoverable strains of a much larger magnitude than previously suspected. Obviously, any technique that measures only a permanent or plastic strain is incapable of obtaining such information concerning recoverable (elastic) deformations.

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[5] "An Experimental Study of the Propagation of Transient Longitudinal Deformations in Elastoplastic Media," by E. J. Sternglass and D. A. Stuart, JOURNAL OF APPLIED MECHANICS, Trans. ASME, vol. 75, 1953, pp. 427–434.

[6] Associate Professor of Materials, Cornell University, Ithaca, N. Y.

Also, the authors have indicated that the dynamic stress-strain curve for annealed 2S aluminum lies considerably above the static curve. In particular, it is stated that the overstress varies from zero to a maximum of 20 per cent of the stress at the highest attainable strain. It seems likely that such an effect is due to the strain rate and, if so, one would expect that the amount of allowable overstress would vary not only with the strain rate but also with the effective impact duration. It would be of interest to know whether such a variation of the amount of overstress with impact duration was observed or investigated. In addition, although it has been believed that face-centered cubic structures do not exhibit the phenomena of "delayed yield," preliminary experiments by H. Sack[7] have shown that annealed copper exhibits such an effect. In view of this fact, the change, if any, of the dynamic stress-strain curve with impact duration is of paramount importance.

Finally, since the von Kármán theory was derived with the assumption that the strain-rate effect is negligible, the writer has some question concerning the significance of the dynamic stress-strain curves obtained by applying the von Kármán theory to an experiment which deviates so widely from the assumptions of the theory. It is believed that a justification of this use of the theory is necessary.

### AUTHORS' CLOSURE

The question raised by J. D. Campbell regarding the accuracy with which the experimental curves shown in Figs. 2 and 3 of the paper can be established may be answered as follows: Some purely elastic impact tests which were performed showed agreement between theoretical and experimental results within ±1.2 per cent. Thus the authors conclude that the accuracies of stress and impact-velocity measurements are ±1.5 per cent or better. The absolute error in the residual-strain measurements is ±0.05 per cent strain or less which is determined by the sensitivity of the device employed to measure the specimen diameter. The additional scatter of the experimental points in Figs. 2 and 3 over and above their uncertainties may be attributed to slight variations in properties between the specimens. The authors feel that the data justify the conclusion that a slight but real difference exists between the two stress versus strain relations designated $A$ and $B$ in Fig. 4.

The modified stress versus strain relations given in Fig. 8 represent the consequences of the simplest hypothesis which the authors have found to be capable of describing the difference between the relations $A$ and $B$ of Fig. 4. At best, this hypothesis (the existence of an individual stress versus strain relation corresponding to each value of the maximum impact strain) can only be a rough approximation to reality.

The wave solution, represented in Fig. 5 of the paper, was constructed on the basis of stress versus strain relation $A$ in Fig. 4. E. H. Lee correctly points out that, if the stress versus strain relations of Fig. 8, which depend upon the maximum impact strain, are accepted, completely separate wave solutions are required for each impact. Constructions to determine such wave solutions were made just to a sufficient extent that the theoretical distances of propagation of the maximum strains (as given in column B of Table 1) could be determined. The authors are of the opinion, in view of the relatively small differences between curve $A$ of Fig. 4 and the curves of Fig. 8 and the amount of labor involved in making the complete construction for a wave solution, that the completion of such individual wave solutions was not worth while.

The values of the elastic limit constant, $\epsilon_0$, and the constant,

[7] Private communication from Professor H. Sack, Department of Engineering Physics, Cornell University, Ithaca, N. Y.

$\alpha$, pertaining to the three stress versus strain relations given in Fig. 8 are listed in the following table:

| Final strain, per cent | $\epsilon_0$ Per cent | $\alpha$ $10^6$ ips |
|---|---|---|
| 2 | 0.045 | $-0.59$ |
| 3.5 | 0.044 | $-0.21$ |
| 5.0 | 0.043 | $-0.115$ |

These values of $\epsilon_0$ correspond to a considerable increase in the elastic limit of the material above the static value. However, no great quantitative significance should be attached to these values. By employing other more complex forms of relations between strain and velocity of propagation than the one used in the paper, the experimental data could be fitted equally well with other values of the elastic-limit strain. To determine the actual elastic-limit strain under impact conditions requires further experimental work in which more sensitive measurements of strains of that order are made. Such measurements are being made currently.

L. E. Malvern's suggestion, that the elevated stress versus strain relation under impact loading might be the result of an increased strain hardening due to greater crystalline disordering under impact conditions, is an interesting one. The difficulty of explaining the uniformity of residual strain near the impact end on the basis of such an hypothesis could be surmounted if it were assumed that the additional strain hardening was a fixed amount (at a given strain) and occurred discontinuously when the strain rate reached a certain critical value.

D. A. Stuart and L. E. Malvern have both inquired about strain versus time relations in the specimen. A few such measurements were made during the investigation, and the records of two of them are presented in the accompanying Figs. 1 and 2. Reliable values of the velocity of propagation as a function of strain cannot be derived from these records, as might be hoped, because the time scale is not sufficiently accurate. This is a result of insufficient frequency-response range of the amplification system employed in making these records.

The theoretical curves indicated in Figs. 1 and 2 were derived from the stress versus strain relation based upon the stress and particle-velocity measurements (curve $A$ in Fig. 4) by means of the wave solution shown in Fig. 5. The difference between the experimental and theoretical values of the final strain indicated in Fig. 9 corresponds to the differences in strain at the appropriate stress between curves $A$ and $B$ of Fig. 4. Thus if theoret-
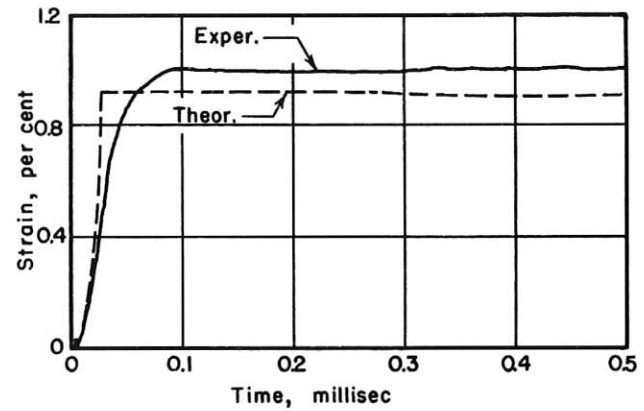


Fig. 1 Strain Versus Time, 0.9 In. From Impact End; Impact Velocity 44.7 FPS
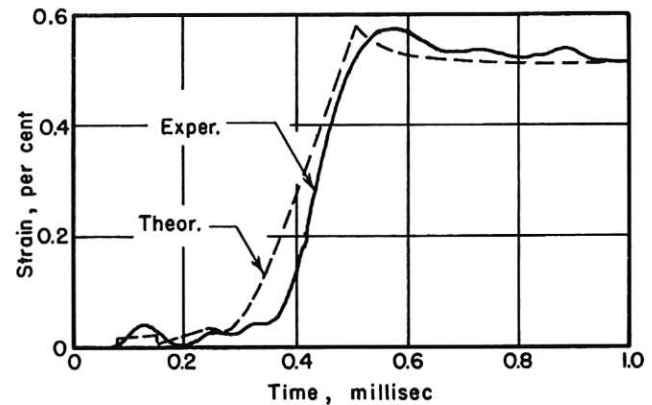


Fig. 2 Strain Versus Time, 16.0 In. From Impact End; Impact Velocity 69.8 FPS

ical strain versus time relations were derived on the basis of the stress versus strain relations given in Fig. 8, this discrepancy in Fig. 9 would disappear.

The authors wish to express their appreciation for the stimulating discussions presented by Messrs. Campbell, Lee, Malvern, and Stuart.