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For the purpose of comparison, the values of the moment on the fixed edge of a square plate which were presented in Table 3 of the paper are listed in the following, together with the corresponding values calculated from the solution obtained by Siess and Newmark:

Location		Clamping moments × $(1/qa^2)$		
x	y	Huang and Conway	Stiles	Siess and Newmark
0.2a	0	−0.0272	−0.0257	−0.0275
0.4a	0	−0.0594	−0.0608	−0.0598
0.6a	0	−0.0692	−0.0684	−0.0691
0.8a	0	−0.0510	−0.0515	−0.0507

Authors' Closure

The authors are grateful to Mr. Veletsos for bringing the work of Siess and Newmark to their attention. The agreement in the values of the clamping moments is most satisfactory.

The Elastic Sphere Under Concentrated Loads[1]

M. M. Frocht.[2] The paper raises several questions regarding the fundamentals in the theory of elasticity. Of particular significance is the conclusion that to the three conditions generally accepted as sufficient for a unique solution a fourth "limit condition" must be added and that failure to include such a fourth condition will, in the case of concentrated loads, lead to pseudosolutions, that is, in essence, to false solutions. This conclusion carries with it the need for a re-examination of the meaning of Saint Venant's principle.

To an experimentalist this paper is significant for an additional reason. It provides another illustration that mathematicians are not infallible and that the ultimate test of a theoretical solution lies in experimental verification.

The agreement between the theoretical results of the authors and the photoelastic results of Frocht and Guernsey are particularly noteworthy because the photoelastic solution is the first complete solution of its kind. The degree of agreement between the two sets of results is indeed remarkable, particularly at the center of the sphere where the difference in the value of (σ_z/σ_0) is about 2 per cent.

On the surface of the sphere, around the equator, where the stresses are relatively small, the agreement is less satisfactory. There the photoelastic value of (σ_z/σ_0) is zero and the theoretical value is 0.30, for $\nu = 0.48$. This aspect of the problem can be investigated by means of a strain-gage test. At the suggestion of the discusser, Messrs. K. P. Milbradt,[3] D. Landsberg,[4] and P. D. Flynn[5] made such a test using an aluminum sphere of 4.87 in. diam for that purpose, Fig. 1, herewith.

Four $^1/_8$-in. SR-4 strain gages were mounted at points A-A, Fig. 2, on the equator. Two gages were set parallel to ϵ_z and the remaining two parallel to ϵ_γ. Tests were made at 28,000 lb, 56,000 lb, 84,000 lb, etc., up to 500,000 lb. Typical strain-load curves for several cycles of loading and unloading are shown in Fig. 3.

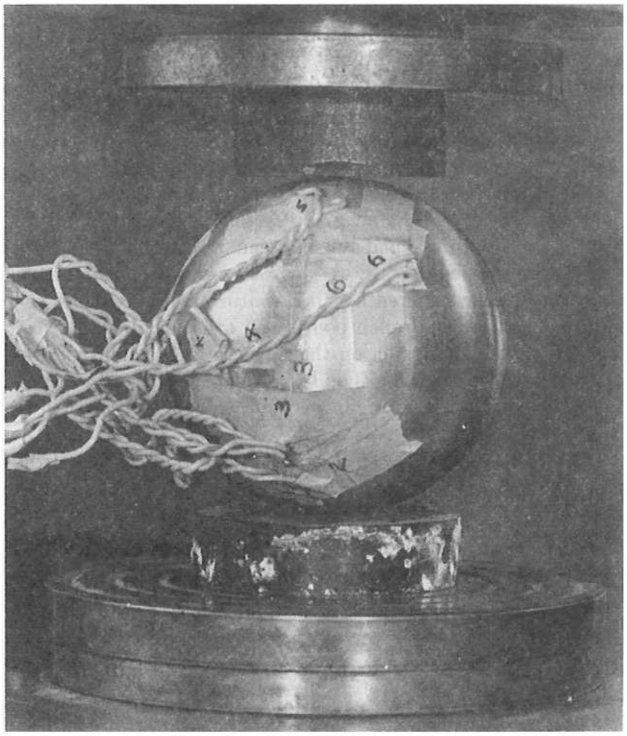

Fig. 1

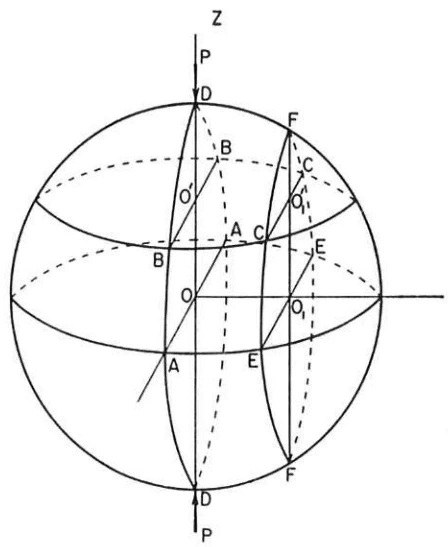

Fig. 2

It will be observed that the stress-strain relation is essentially linear. This linearity extended up to 280,000 lb, and the deviations from linearity at higher loads were small. It also will be noted that recovery was somewhat incomplete in the first stage of each new load. The meaning of this initial set is at present not clear. Perhaps it may be connected with the development of small plastic zones in the region of the applied load.

The stresses were based on the mean recoverable strain. The error resulting from neglecting the initial set is believed to be small.

The final results are summarized in Table 1 of this discussion. The gages thus show that in the equatorial region there exist compressive σ_z stresses of a magnitude of $0.25\sigma_0$ and transverse

[1] By E. Sternberg and F. Rosenthal, published in the December, 1952, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 74, pp. 413–421.

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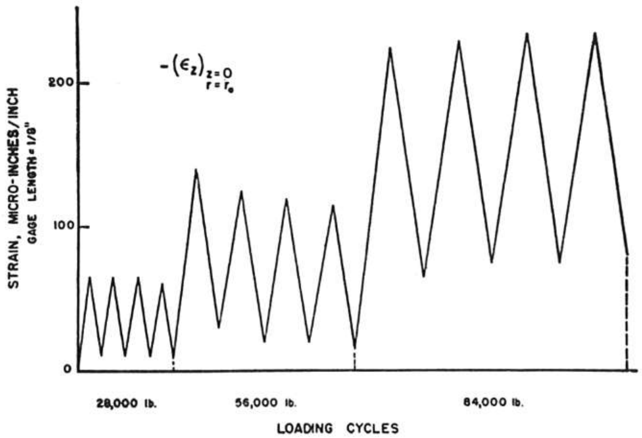

FIG. 3

tensile stresses $0.43\sigma_0$. The corresponding theoretical values for $\nu = 0.32$ are $0.24\sigma_0$ and $0.44\sigma_0$, respectively. The strain-gage tests thus corroborate the analytical conclusions of the authors.

The reason for the small photoelastic error is believed to lie in the thermal stresses developed during the freezing cycle. A discussion of these must be deferred for another occasion.

TABLE 1 RESULTS OF TESTS

	Stresses at $z = 0, r = r_0$			
	σ_z/σ_0	σ_γ/σ_0	ν	Investigator
Theoretical....	−0.24	0.44	0.32	Sternberg and Rosenthal
Strain gage....	−0.25	0.43	0.32	Milbradt, Landsberg, and Flynn
Photoelastic...	0	0.68	0.48	Frocht and Guernsey

N. J. HOFF.[6] This excellent paper contains a result of much wider significance than the title implies. It shows that the application of a self-equilibrating system of indefinitely large forces to a vanishingly small part of the surface of an elastic body gives rise to strains of not negligible magnitude at finite distances from the part. This seems to contradict Saint Venant's principle. It is the purpose of this discussion to explain the discrepancy by showing that Saint Venant's principle does not necessarily apply in problems involving mathematical singularities.

In so far as the empirical foundation of the principle is concerned there is not much possibility for an argument because no true singularities can arise in nature and no practical concentrated load can cause stresses much in excess of the yield stress of the material. If one re-examines Goodier's[7] strain-energy analysis of the principle, he must conclude that the odd behavior of the stresses discovered by the authors is not really unexpected. Goodier showed that that part of the elastic body, in which the order of magnitude of the stress is the same as the order of magnitude of the applied self-equilibrating tractions, is of the order of magnitude of the linear dimensions of the loaded part of the body. It does not follow from this statement that stresses of a finite magnitude cannot be found at any finite distance from the loaded part if the loading comprises tractions which are indefinitely large, as will now be shown by means of a modification of Goodier's argument.

Let the authors' problem be modified by replacing each concentrated load by statically equivalent distributed loads, acting upon small parts of maximum linear dimensions L_0, of the surface of the sphere. Because of the axial symmetry of the problem, $L_0/2$ may suitably be chosen as the radius of a circle, whose center is the point of attack of the original concentrated load, and which encloses the loaded part. The maximum absolute value of the applied surface tractions in this system S_0 will be denoted by s_0. The system will be so chosen that s_0 is approximately equal to the elastic limit of the material. Let it also be assumed that all the stresses and strains caused in the sphere by the system S_0 are known.

A second system S_1 consists of distributed surface tractions of maximum absolute value s_1 acting upon two small parts, each of maximum linear dimension L_1, of the surface of the sphere. $L_1/2$ will be chosen as the radius of a circle enclosing the loaded part whose center is the point of attack of the original concentrated load. Over each small part the loading is again statically equivalent to the original concentrated load. Obviously, the stresses and strains caused in the sphere by system S_1 can be determined by superposition of the stresses and strains caused by system S_0 and those caused by the combined system $S_2 = S_1 - S_0$. System S_2 contains self-equilibrating surface tractions distributed over two small areas of maximum linear dimension L_0 or L_1, whichever is greater. The maximum absolute value of the surface traction is $s_2 \leq s_0 + s_1$.

It is not unreasonable to consider the stresses and strains caused by system S_0 as the standard solution, and to define that singly connected region of base $(\pi/4)L_0^2$ of the sphere which includes all the points at which the maximum stress is equal to or greater than s_0/n as the region affected by the details of the load distribution. The value of n is to be chosen from considerations of accuracy; $n = 2$ defines a region of large disturbances while outside the disturbed region corresponding to $n = 100$ the effect of the details of the load application is probably always negligible for engineering purposes.

The strains caused by system S_2 in the loaded region are not greater than s_2/G, where G is the shear modulus of the material. If one infinitesimal area of the loaded surface is fixed in space, the relative displacement of any point of the loaded surface with respect to the reference area is not greater than s_2L_0/G if $L_0 > L_1$ provided the effect of the rotations of the elements is neglected. If this effect is taken into account in the authors' problem, the relative displacement cannot be greater than ms_2L_0/G, where m is a positive number of the order of magnitude of unity. As displacement and load may be of the same sign over part of the loaded surface and of opposite sign over the rest, the total work done by the loads during a proportional loading is

$$W < (1/2)(\pi/4)ms_2^2L_0^3/G$$

The singly connected disturbed region just defined will now be approximated by a sphere of diameter D included in the sphere analyzed by the authors and having a common tangent plane with it at the point of attack of the original concentrated load. Inside this sphere the maximum absolute value of the stress is not everywhere greater than or equal to s_0/n, but there are necessarily subregions where this is true. Nevertheless, if n is chosen sufficiently large, s_0/n is much smaller than s_2 and the strain energy stored in the disturbed region

$$U > (\pi/24)\,(s_0^2/n^2G)D^3$$

[6] Head of Department of Aeronautical Engineering and Applied Mechanics, Polytechnic Institute of Brooklyn, Brooklyn, N. Y. Mem. ASME.

[7] "A General Proof of Saint-Venant's Principle," by J. N. Goodier, *Philosophical Magazine*, Series 7, vol. 23, April, 1937, p. 607; Supplementary Note on "A General Proof of Saint-Venant's Principle," *Philosophical Magazine*, Series 7, vol. 24, August, 1937, p. 325; "An Extension of Saint-Venant's Principle, With Applications," *Journal of Applied Physics*, vol. 13, March, 1942, p. 167. See also "Theory of Elasticity," by S. Timoshenko and J. N. Goodier, second edition, McGraw-Hill Book Company, Inc., New York, N. Y., 1951, p. 150.

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It follows then from the principle of the conservation of energy

$$U = W$$

that

$$D < (3\ mn^2)^{1/3}(s_2/s_0)^{2/3}L_0$$

It is of importance that in this equation the reference length L_0 is the diameter of the circle enclosing that loaded region in which the maximum surface traction is approximately equal to the elastic limit of the material. It follows from the equation that the diameter D of the disturbed region is smaller than $(3mn^2)^{1/3}$ times L_0 if $s_2 = s_0$. When the two systems S_0 and S_1 differ little, s_2 is much smaller than s_0 and the disturbed region is small. On the other hand, when S_1 is a concentrated load over each area, $L_1 \rightarrow 0$, both s_1 and s_2 are indefinitely large, and the disturbed region may extend over a distance which is the product of L_0 by a very large number. Hence the use of the somewhat artificial, although most useful, mathematical concept of a concentrated load may, in some cases, lead to results in disagreement with experimental evidence.

The argument given implies that no independent nuclei of high stress exist inside the elastic body detached from the high-stress region connected with the loaded surface. Their existence in a homogeneous elastic body would violate the complementary energy principle.

The entire discussion presented referred to a three-dimensional solid body. In a statically determinate framework the forces are uniquely determined by the equilibrium conditions, and self-equilibrating systems of loads applied to a small number of joints may cause large stresses at distances from the loads which are large compared to the maximum distance between any two applied loads. When a large number of redundant bars is contained in the framework, Saint Venant's principle holds reasonably well, as was shown by the writer in an earlier publication.[8]

Finally it should be mentioned that in an investigation dealing with concentrated loads R. von Mises[9] showed that Saint Venant's principle does not, in general, hold for bodies of finite dimensions.

E. H. Lee.[10] The presentation and discussion of this paper emphasized the remarkable differences in the stresses away from the point of concentrated loading given by solutions involving different singularities, in apparent contradiction to the usual interpretation of Saint Venant's principle. It was suggested in the discussion that in the practical application of a problem of this type, such differences could be generated by the details of the contact conditions, such as friction. It would seem that in the case of contact over a small area, such additional surface tractions will always produce negligible changes in stress away from the point of contact, so that the solution presented will apply independently of the details of loading.

The reason for this statement is that such traction systems will be self-equilibrating, for example, in the case of friction a ring of radial shear traction at the surface, and can be considered as force doublets. The force magnitudes will be of the order of the applied normal force and in forming a doublet these will be multiplied by a dimension of the order of the extent of contact. The analysis of the stresses due to such a doublet will involve a smaller order of stress than that corresponding to the resultant applied load. The same also will be true of additional radial forces if the load is applied as pressure in an indentation. In considering the representation of these additional forces as doublets, the limit is taken of the product of force and separation to remain constant, while the force magnitude increases indefinitely, and the separation decreases to zero. However, this product is prescribed by the applied force and the contact dimension, so that the strength of the doublet will be small for small contact area, and the corresponding additional stresses will be negligible compared with those associated with the solution under discussion. The writer wishes to acknowledge help in clarification of these points during conversation with Dr. J. N. Goodier.

Authors' Closure

The authors appreciate the interesting comments of Professors Frocht, Hoff, and Lee. The additional experimental corroboration of the theoretical results is, of course, gratifying. Professor Frocht's remark that the "limit condition" must be *added* to the usual three necessary conditions in order to assure a unique solution to a concentrated-force problem, was evidently prompted by a statement in the paper, the wording of which is apt to be misleading. Actually, the requirement that the solution coincide with the appropriate limit of the solution corresponding to continuous surface tractions which are arbitrarily distributed over regions surrounding the points of application of the prescribed concentrated loads, by itself assures uniqueness and guarantees the satisfaction of the three necessary conditions referred to. The limit condition may thus be regarded as a definition of what is meant by the solution to a problem involving concentrated surface loads, in analogy to Kelvin's definition through a limit process of the concept of an interior concentrated force. The particular choice of these definitions is natural both on theoretical and physical grounds; the usefulness of the idealization of "concentrated forces" so specified ultimately rests on experimental verification. The photoelastic results of Frocht and Guernsey, as well as Professor Lee's observations, are relevant in this connection. It should be mentioned that the present definition of concentrated surface loads is consistent with Kelvin's definition of interior concentrated forces, as is readily confirmed.

A good deal of the discussion appears to center around the result that a solution appropriate to self-equilibrated singularities at the poles of the sphere, and otherwise clearing the surface from tractions, yields finite stresses at the center of the sphere. The authors agree with Professor Hoff that this result does not contradict a rigorous statement of Saint Venant's principle although they would base this statement on a different type of argument.

On this occasion reference should be made to a simultaneous treatment of the problem under discussion by C. Weber.[11] Weber, by different means, obtained the solution to the problem in the series representation designated by S_L in our paper (see Equation [58]). His solution therefore suffers from the convergence deficiencies mentioned in the discussion of S_L. As pointed out in our paper, the singularities inherent in the problem require special and separate attention. Moreover, Weber's paper unfortunately contains two errors which are carried through a major portion of the development and are responsible for the incorrectness of his final stress formulas: the leading term in the denominator of his equation [11] should read $(n-1)\ (n-2)$ instead of $(n-1)^2$, whereas in his equation [23] the first factor in the numerator of the second fraction should read $(2n+5)$ instead of $(2n+3)$. In conclusion, the authors wish to call attention to a

[8] "The Applicability of Saint-Venant's Principle to Airplane Structures," by N. J. Hoff, *Journal of the Aeronautical Sciences*, vol. 12, October, 1945, p. 455.

[9] "On Saint-Venant's Principle," by R. von Mises, Bulletin of the American Mathematical Society, vol. 51, 1945, p. 555.

[10] Professor of Applied Mathematics, Brown University, Providence, R. I. Mem. ASME.

[11] "Kugel mit Normalgerichteten Einzelkraften," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 32, no. 6, 1952, pp. 186–195.

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paper by J. L. Synge,[12] in which the closed formulas for the stresses at the center of the sphere were derived directly by altogether different considerations.

On Longitudinal Plane Waves of Elastic-Plastic Strain in Solids[1]

WERNER GOLDSMITH.[2] The author is to be congratulated on a significant contribution in the field of wave propagation in solids, particularly in view of the applicability of the paper to a proper understanding of the mechanism of resistance and failure of metals subjected to contact explosions. Several interesting points have been raised by this presentation.

In the conversion of ordinary tensile and compressive stress-strain curves to those appropriate for longitudinal waves exhibiting no transverse deformation, it is rather surprising to note that the effective yield-point stress—denoted by the symbol 1 in Fig. 2 of the paper—has been increased nearly 100 per cent over the yield point in simple tension for the 24S-T aluminum-alloy specimen. A similar calculation by means of Equation [10] of the paper indicates that materials with a Poisson's ratio $\mu = 0.45$, such as lead, would possess a yield point under the action of plane longitudinal waves equal to 550 per cent of that observed in simple tension tests. These observations lead to the conclusion that the normal elastic range of a material may be extended considerably by imposing external constraints on the motion of the system. The lateral inertia exhibited by a material subjected to high transient loads produces an identical constraining effect and consequent increase in the instantaneous dynamic yield point. This phenomenon is independent of the normal strain-rate sensitive behavior of the material proper, which is also responsible for an apparent increase in the dynamic yield point, as discussed by the author in the last paragraph and further described in reference (4) of the paper.

The conditions of the present analysis involve the most severe type of restriction on any actual particle motion of the system. Thus the yield point for plane longitudinal waves, derived from Equation [10] as

$$\sigma_x = \frac{1-\mu}{1-2\mu}\,\sigma(\epsilon_x{}^p)$$

presumably represents the upper limit of a specimen subjected to any type of constraint upon its deformation, while the yield point in simple tension may be considered as the lower limit of this parameter. Any system with different constraints should exhibit a yield point intermediate between these two extremes.

In the experiments of references (1) and (7) of the paper, the geometry employed is such that the propagation of strain occurs by means of spherical waves of dilatation rather than by plane longitudinal waves. A comparison of the two systems of propagation shows that in both cases the velocity of elastic waves is given by

$$c_e = \left[\frac{E}{\rho}\,\frac{(1-\mu)}{(1+\mu)(1-2\mu)}\right]^{1/2}$$

Utilizing an idealized tensile stress-strain curve consisting of two straight lines with moduli E and E_1, respectively, and an assumption similar to that represented by Equation [7] of the paper, the writer has shown that the velocity of propagation of spherical plastic waves is given by

$$c_p = \left[\frac{3K(1-2M)}{\rho}\right]^{1/2}$$

where

$$M = \frac{\left(1-\frac{E_1}{E}\right)(1-2\mu)+3\mu}{3(1+\mu)}$$

This differs by the correction factor $2M$ from the usual propagation velocity $c_p = [3K/\rho]^{1/2}$ cited by reference (7) of the paper, which was numerically closely substantiated by the author. However, the general treatment required for spherical waves involves considerably greater mathematical complexities than for the case under consideration, particularly in view of the attenuating effect of the divergence of spherical waves. This results in a decrease of the stress amplitudes of both elastic and plastic waves with increasing radius, in addition to the destruction of plastic-wave amplitude due to annihilation by unloading waves. The writer is endeavoring to evaluate some quantitative results along these lines.

J. S. RINEHART.[3] This paper represents a valuable step forward toward our understanding and appreciating the factors which control scabbing of impulsively loaded metal plates. The author mentions (*a*) effect of variable compressibility, (*b*) effect of temperature variations, and (*c*) time and rate effects. Another important factor is the changes which are wrought by energy absorption due to internal friction. In general, this absorption will cause the pulse to be attenuated and to become longer as it moves through the body. It is to be remembered that the momentum of the disturbance must be conserved even though energy is dissipated. A further factor which will contribute to the lengthening of the pulse is the variations of Poisson's ratio and bulk modulus with pressure. These effects have been treated in the case of steel by Koehler and Seitz.[4] The effect of decay is strikingly illustrated by some examples of scabbing of flat plates that were explosively loaded in the manner shown in Fig. 1 of this discussion.

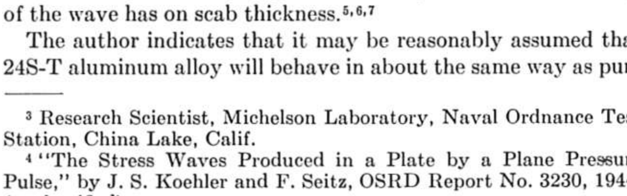

FIG. 1

The two curves in Fig. 2 show clearly the effect that decay of the wave has on scab thickness.[5,6,7]

The author indicates that it may be reasonably assumed that 24S-T aluminum alloy will behave in about the same way as pure

[12] "Upper and Lower Bounds for the Solution of Problems in Elasticity," Proceedings of the Royal Irish Academy, vol. 53, series A, no. 4, 1950, p. 41.

[1] By D. S. Wood, published in the December, 1952, issue of the JOURNAL OF APPLIED MECHANICS, Trans. ASME, vol. 74, pp. 521–525.

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[3] Research Scientist, Michelson Laboratory, Naval Ordnance Test Station, China Lake, Calif.

[4] "The Stress Waves Produced in a Plate by a Plane Pressure Pulse," by J. S. Koehler and F. Seitz, OSRD Report No. 3230, 1944, (unclassified).

[5] "Investigations of Fracture Produced by Transient Stress Waves," by H. Kolsky and A. C. Shearman, Research 2, 1949, p. 384.

[6] "Behavior of Metals Under High and Rapidly Applied Stresses of Short Duration," by J. S. Rinehart, NavOrd Rept. No. 1138, September 1949.

[7] "Explosive Forces Widen Metallurgical Studies," by J. S. Rinehart, *Steel*, Nov. 20, 1950, p. 98.

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