# Discussion

## Stresses and Deformations of Toroidal Shells of Elliptical Cross Section[1]

WALTER WUEST.[2] The Bourdon tube, invented in 1845 by the German engineer Schinz, has gained a wide field of application for industrial-pressure measurements. For a long time the tubes were manufactured quite empirically. The originally elliptical form of the cross section was changed to a form consisting of two half circles and two straights. During the last war the latter form was investigated by thorough experiments in the Schäffer & Budenberg spring laboratory (Germany). The measured values were found in agreement with theoretical results of the writer, who simplified and improved M. Tueda's theory of Bourdon tubes. Some experimental results also were gained with Bourdon tubes of elliptical cross section.

It is now interesting to compare these experimental results with the theory presented in this paper. The theoretical results for the pressure sensitivity show about double the value of the measurements. Also, taking into account that it is difficult to retain the exact elliptical form during the manufacturing process, this disagreement seems to be very high. Further measurements are needed to clarify this situation. Such measurements would be very interesting, because the results of the authors' theory indicate an extraordinarily great influence of the form of the cross section.

The general theory of thin-walled Bourdon tubes is quite complicated. Apart from the elastic properties of the material the elastic behavior of such tubes depends for a given form of cross section mainly on the following two geometrical parameters:

1 The ratio $\alpha$ of the axes of the cross section.

2 The parameter $\Lambda = c^2/ah$ with $c$ = great semiaxis of the cross section, $a$ radius of curvature, $h$ thickness of the wall.

[1] By R. A. Clark, T. I. Gilroy, and E. Reissner, published in the March, 1952, issue of the JOURNAL OF APPLIED MECHANICS, Trans. ASME, vol. 74, pp. 37–48.

[2] Max Planck Institut für Strömungsforschung, Göttingen, Germany.

There are two special cases for which the solution is rather simplified:

(a) $\Lambda$ small (in practice $\Lambda < 1$), arbitrary $\alpha$.
(b) $\alpha \to 0$, arbitrary $\Lambda$.

The first limiting case may be called "mean pressure tube," for this simplification is only justified for tubes of mean pressure range (lying in practice approximately between 700 and 7000 psi). The elementary theory of mean pressure tubes has often been treated. C. B. Biezeno and J. J. Koch[3] have discussed a numerical method for determining arbitrary cross sections. By an iterative process they extended the range of validity of the elementary theory up to approximately $\Lambda = 3$; but in practice values up to $\Lambda = 50$ happen.

Also, in the other limiting case $\alpha \to 0$ the theory is simplified greatly. A detailed analysis will be treated in a paper by the writer to be published in the *Ingenieure Archiv*.[4] If the cross section is supposed to be of double symmetry and given by

$$y = \pm bf(x)$$

with $f(x)$ an odd function in $x$, the analysis leads to the following differential equation for the normal deformation $w^*$

$$w^{\text{iv}}(\xi) + 4n^4 w = 4n^4[\alpha\omega f(\xi) - A] \quad [1]$$

with $w = w^*/c$, $\xi = x/c$, $\omega = \Delta a/a$

$$n^4 = 3(1 - \nu^2)c^4/a^2h^2$$

$$A = 3(1 - \nu^2)c^3p/n^4h^3E$$

The boundary conditions are

$$w = w' = 0 \text{ for } \xi = \pm 1$$

and the general solution is given by

[3] "On the Elastic Behaviour of the So-Called Bourdon Pressure Tube," by C. B. Biezeno and J. J. Koch, Proceedings, Amsterdam, vol. 44, 1940, pp. 779–786, and 914–920.

[4] "The Influence of the Form of Cross Section on the Behaviour of Bourdon Tubes" (in German), by Wuest, *Ingenieure Archiv*, vol. 20, 1952, pp. 116–125.

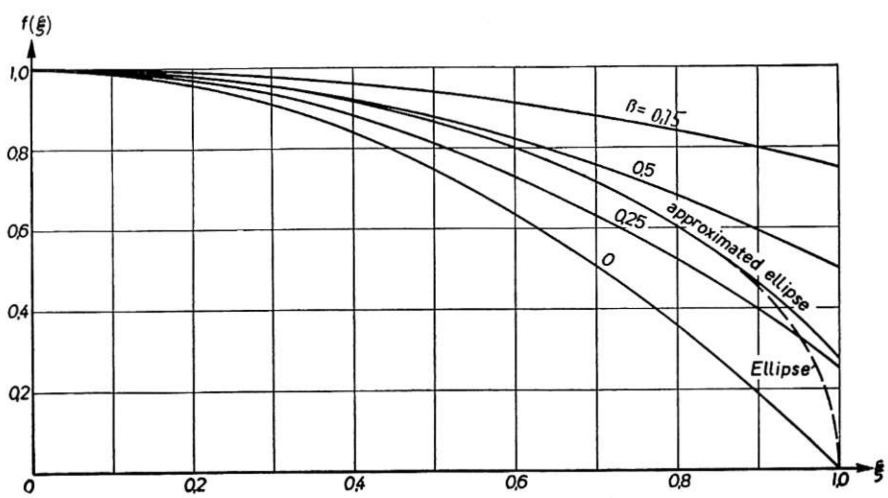


FIG. 1

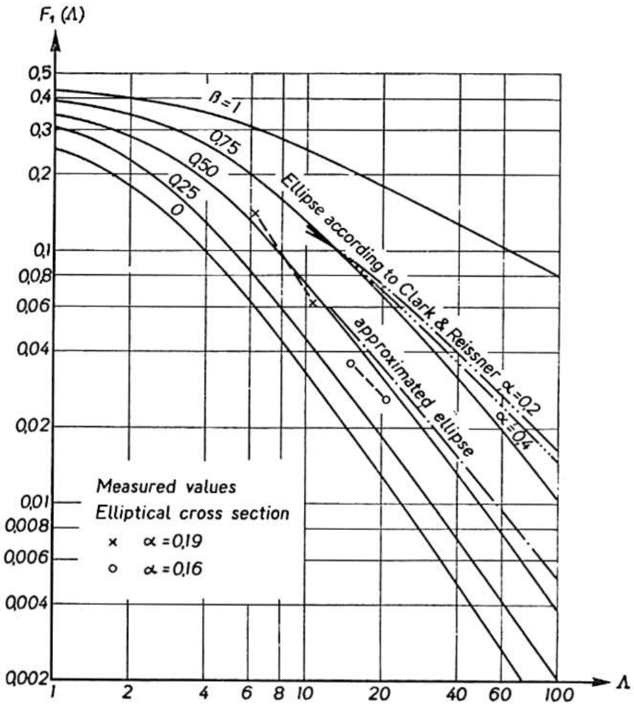


Fig. 2

$$w = A(1 + C_1 \cosh n\xi \cos n\xi + C_2 \sinh n\xi \sin n\xi) + n\alpha\omega \int_0^\xi \overline{w}(\xi - \xi')f(\xi')d\xi' \quad \ldots [2]$$
$$\text{with } \overline{w}(\xi) = \cosh n\xi \sin n\xi - \sinh n\xi \cos n\xi$$

Explicit solutions are possible for $f(\xi)$ given by a polynom, exponential, or trigonometric functions.
The pressure sensitivity is given by

$$\Delta a/a = \frac{1}{\alpha} \frac{\int_0^1 wfd\xi}{\int_0^1 xf^2d\xi} \ldots\ldots\ldots\ldots\ldots [3]$$

Fig. 1 of this discussion shows a family of cross sections given by $f(\xi) = 1 - \beta\xi^2$. $\beta = 0.5$ is a very rough approximation of an ellipse. A better approximation is given in the figure by a polynom of 6th degree (approximated ellipse). The evaluation of Equation [3] of this discussion leads to

$$\Delta a/a = F_1(1 - \nu^2)c^3p/\alpha h^3E \ldots\ldots\ldots\ldots [4]$$

In general, $F_1$ is a function of $\Lambda$ and in minor dependence also of $\alpha$. Here only the value lim $F_1(\alpha, \Lambda)$ for $\alpha \rightarrow 0$ is investigated and shown in Fig. 2 of this discussion, for the forms of cross section of Fig. 1. The values of the authors' theory, interpolated for $\alpha =$ 0.2 and 0.4 and the measurements mentioned are represented by dotted lines. The measured values lie near the limit curve of the approximated ellipse.

Also, other forms of cross section have been investigated by this method. The approximated analysis concerns also the bending of curved tubes and the stresses. An interesting problem is also the protection of the Bourdon spring from damage by excessive pressures and the behavior of dynamical differential manometers. These problems, too, have been investigated by approximate analysis, i.e., for the limiting case $\alpha \rightarrow 0$.

### Authors' Closure

Dr. Wuest's comments on the portion of our paper having to do with the sensitivity of Bourdon tubes are of considerable interest and the paper which he has just published[4] represents an important contribution to the subject.

Dr. Wuest's calculations (see Figs. 1 and 2) show that for sufficiently large values of $\Lambda$ the sensitivity of the tube depends very greatly on the form of its cross section. Dr. Wuest also refers to previously unpublished experimental results concerning tubes with elliptical cross section. We feel that these experimental results are not incompatible with our calculations in view of the important effect which relatively small changes in shape are seen to have and in view of the discusser's statement[4] "Die bei den Messungen verwandten Röhrenfedern sind zunächst mit elliptischen Ziehdornen gezogen worden. Doch besteht immerhin die Möglichkeit, dass bei dem 'Rollvorgang,' bei dem das ursprünglich grade Rohr eine kreisförmige Krümmung erhält, die elliptische Form nicht streng erhalten blieb."

As we see it, Dr. Wuest's calculations are based on the assumptions that

$$\Lambda \text{ finite}, \alpha \rightarrow 0 \ldots\ldots\ldots\ldots\ldots\ldots [1]$$

and there seems to be an upper limit, at least when $\beta \neq 0$, for the admissible values of $\Lambda$ for which the boundary conditions $w'(\pm 1) = 0$ result in a true picture of what happens. This is in accordance with the fact that as $\Lambda$ increases the shape of the cross section and what happens in the immediate vicinity of the ends of the major axis is of major importance.

Our own calculations for the elliptical cross section were for

$$\alpha \text{ finite} \ldots\ldots\ldots\ldots\ldots\ldots\ldots [2]$$

and we did not offer results which were intended to be valid when $\alpha \rightarrow 0$. On the other hand, our results were for $\Lambda$ finite where we used trigonometric series *and* for $\Lambda \rightarrow \infty$ where we used a method of asymptotic integration. We think it would be interesting to consider the problem further, in particular with reference to what happens when simultaneously

$$\alpha \rightarrow 0, \Lambda \rightarrow \infty \ldots\ldots\ldots\ldots\ldots\ldots [3]$$

# Gas Cooling of a Porous Heat Source[1]

Edward Adams Richardson.[2] The writer, and his partner, hold patents on thermal insulation,[3] chemical apparatus,[4] and furnaces,[5] involving the use of permeable bodies with fluid flowing between two faces at different temperatures. In view of such inventions, it was obvious to the partners that such permeable bodies with heat liberation in the mass thereof could be cooled by methods such as those proposed.

We find the work done by the author of interest. However, it seems well to discuss certain aspects of this paper.

The significance, or necessity, for certain of the assumptions made deserves consideration. Though customarily made, the first assumption to the effect that the permeable body and permeating fluid are at the same temperature is not essential. If $H$ is the rate of heat flow between solid and fluid in Btu per cu

[1] By Leon Green, Jr., published in the June, 1952, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 74, pp. 173–178.

[2] Partner of Edward (Nellie), and George Richardson of Bethlehem, Pa. Mem. ASME.

[3] U. S. Patent No. 2,215,532, Method and Apparatus Relating to Insulated Vessels and Structures of Great Variety, issued to E. A. Richardson, September 24, 1940.

[4] U. S. Patent No. 2,235,644, Process and Apparatus for Effecting Chemical Reactions Involving a Melt and a Gas-Like Body, issued to E. A. Richardson, March, 18, 1941.

[5] U. S. Patent No. 2,311,350, Method and Apparatus for Controlling Combustion, issued to E. A. Richardson, February 16, 1943.