Wherever the inequality in Equation [3] herewith is valid, it must satisfy Equation [1]. This problem is an exceedingly difficult one analytically, and only a very few special cases have been solved.[4] Therefore, if it were possible to construct an electrical analog for this case, it would be a valuable contribution to elastic-plastic theory. This would involve a nonlinear element, which would, in effect, put a "ceiling" on the current that could be carried by any element. Current which would violate this ceiling must then be "rerouted" via other elements in such a way as to preserve the continuity conditions.

Author's Closure

The author wishes to thank Dr. Barton for the numerical verification of the analog-computer results. In the author's opinion, the reason for the lack of agreement between the experimental value for the lowest mode frequencies and the values calculated by finite differences lies in a crude and fortunately avoidable approximation in the distribution of inertias. In the author's paper and presumably in Dr. Barton's calculations, unit masses are connected to interior points; $^1/_2$ unit masses are connected to edge points and $^1/_4$ unit masses are connected to corner points. If instead a distribution is adopted that is statically equivalent to the distributed mass load, better results will be obtained. The square cantilever plate problem has recently been repeated using a finer-mesh (5 cells on a side). The computed frequency is $0.542 \dfrac{1}{a^2}\sqrt{D/m}$ which compares well with Dr. Barton's experimental result which is $0.546 \dfrac{1}{a^2}\sqrt{D/m}$.

It is probably not practical to use the analog computer to solve the set of equations resulting from an application of the Ritz method. In any event the Ritz method does not produce good values for stresses while in the finite difference method the calculated stresses satisfy statics exactly.

The author wishes to point out to Professor Hodge that the problem of constructing an electric analogy for the elastic plane-strain problem was investigated by G. Kron[5] a few years ago. His plane-strain network was derived as a special case of a network representing a three-dimensional elastic field. Network analyzer solutions were presented in a companion paper.[6]

One form of the plane-strain circuit is similar to the network shown in the author's paper with the W-circuit removed.

The construction of an analog for the elastic-plastic plane-strain problem would be difficult. If the yield inequality were imposed directly on the stress components

$$|\sigma_x| < K_1 \qquad |\sigma_y| < K_2 \qquad |\tau_{xy}| < K_3$$

these conditions could be handled by means of current limiters employing crystal diodes. Unfortunately the condition is imposed on the principal stress as shown in the foregoing Equation [3], which condition cannot easily be imposed on an electric circuit.

[4] "Plane Elastic-Plastic Problem: Plastic Regions Around Circular Holes in Plates and Beams (Russian)," by L. A. Galin, *Prikladnaia Matematika i Mekhanika*, vol. 10, 1946, pp. 365–386; "An Elastic-Plastic Problem With a Nonbiharmonic Plastic State (Russian)," by O. S. Parasyuk, ibid., vol. 13, 1948, pp. 367–370.

[5] "Equivalent Circuits of the Elastic Field," by G. Kron, Journal of Applied Mechanics, Trans. ASME, vol. 66, 1944, pp. A-149–161.

[6] "Numerical and Network Analyzer Solution of the Equivalent Circuits for the Elastic Field" by G. Carter, Journal of Applied Mechanics, Trans. ASME, vol. 66, 1944, pp. A-162–167.

The Propagation of Longitudinal Waves of Plastic Deformation in a Bar of Material Exhibiting a Strain-Rate Effect[1]

D. S. Wood.[2] The author has assumed a certain flow law in which the stress depends upon strain rate as well as strain, in an attempt to obtain a better agreement between the theory of plastic strain waves and experimental observations of these waves. This is a good example of the type of procedure which must be followed in order to determine the mechanical properties of materials at the very high strain rates reached in impact tests. The particular flow law assumed by the author leads to improved agreement between theory and experiment in one respect, but, unfortunately, one of the most prominent features of the experimental results is lost, namely, the region of constant plastic strain near the impact end. The writer has the feeling that this disagreement is associated with the fact that the assumed flow law requires purely elastic response of the material to instantaneous finite changes of stress. This implies that the correct flow law must be such that instantaneous finite changes of stress are, in general, accompanied by instantaneous finite changes of plastic strain. A flow law of this type is

$$\sigma = h(\epsilon) - R(\dot\epsilon)[h(\epsilon) - f(\epsilon)]$$

where

$f(\epsilon)$ represents the static stress-strain relation
$h(\epsilon)$ represents a stress-strain relation for instantaneously applied stresses
$R(\dot\epsilon)$ is a function of the strain rate such that
$R(0) = 1$, and $R(\infty) = 0$

With regard to the results of compressive impact tests on lead (20), referred to by the author, it should be noted that those specimens which did not show a region of constant plastic strain near the impact end exhibited mushrooming at the impact end; that is, marked lateral flow occurred at the impact surface. For this reason, it appears that any theory properly describing these particular experimental results must be a two-dimensional rather than a one-dimensional theory. Hence, it is difficult to see how a theory of the type presented by the author can describe these results properly.

E. H. Lee.[3] The numerical integration, the results of which are given in this paper, illustrates the length and complication of a wave analysis for a stress-strain relation involving a strain-rate dependence. Although the assumed, seemingly reasonable, relationship does not reproduce all the experimental results, it is valuable in emphasizing features of the stress, strain, strain-rate relation required.

The remarkable requirements called for by the experimental results cited in references (18) and (19) of the paper, have not, it is believed, been generally appreciated. After impact a region of uniform permanent strain is obtained, which comprises material which has been subjected to the maximum stress throughout the duration of impact, and material which has only instantane-

[1] By L. E. Malvern, published in the June, 1951, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 73, pp. 203–208.

[2] Assistant Professor of Mechanical Engineering, California Institute of Technology, Pasadena, Calif. Jun. ASME.

[3] Professor of Applied Mathematics, Brown University, Providence, R. I. Mem. ASME.

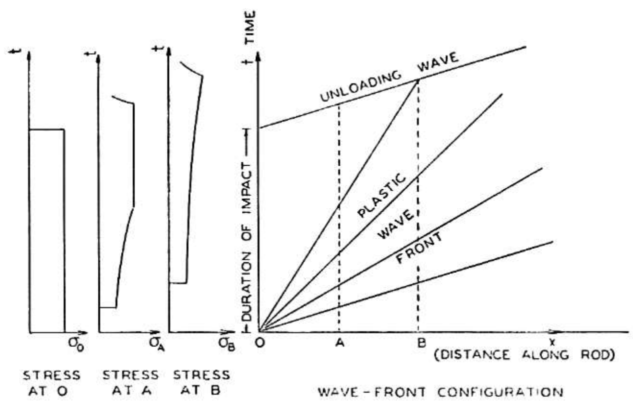

Fig. 1

ously been subjected to this stress. This condition is shown in Fig. 1, herewith, which illustrates approximate loading cycles at different sections of the rod. To a high degree of accuracy, uniform strain is obtained throughout the extent OB of the travel of the wave of maximum stress, and a stress, strain, strain-rate relation must be found which gives the same plastic strain for this widely different range of loading histories.

Author's Closure

The author wishes to thank Professors Wood and Lee for their pertinent discussion of the paper.

Perhaps the flow law should permit some instantaneous plastic response to applied stress, as suggested by Professor Wood; but the author doubts that a law of the particular form suggested would predict the region of constant strain near the impact end, since the factor $R\,(\dot\epsilon)$ would not affect the maximum strain attained at different cross sections in the same degree because the strain-rate history varies greatly from one section to another near the impact end, as was pointed out by Professor Lee. It almost seems that the multiplier R should be a function of the velocity with which the impact end is struck instead of the strain rate in order to predict the constant-strain region, but such a dependence on velocity does not seem reasonable.

The author agrees with Professor Wood that a two-dimensional theory would be needed to describe properly the deformation in lead bars under compressive impact, when mushrooming occurs at the impact end. A law of the type used by the author, which predicts continued flow, could at best only account for the beginning of mushrooming.

Shakedown in Continuous Media[1]

P. G. Hodge, Jr.[2] Professor Symonds is certainly to be congratulated on having extended the concept of shakedown to the important case of continuous media. It is perhaps of some interest to notice that the inequality Equation [25] of the paper, obtained as a necessary condition for shakedown in a particular example, can, in fact, be generalized. To this end, let s_{ij}^* and s_{ij}^{**} be any two admissible elastic-stress states. Then, a necessary condition for shakedown is

$$\left(\frac{s_{ij}^{**} - s_{ij}^{*}}{2}\right)\left(\frac{s_{ij}^{**} - s_{ij}^{*}}{2}\right) \leq 2k^2 \quad\ldots\ldots\ldots [1]$$

[1] By P. S. Symonds, published in the March, 1951, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 73, pp. 85–89.

[2] Assistant Professor of Mathematics, University of California, Los Angeles, Cal. Jun. ASME.

For assume that the Inequality [1] is false, then

$$s_{ij}^\circ s_{ij}^\circ > 8k^2 \quad\ldots\ldots\ldots [2]$$

where the notation $s_{ij}^\circ = s_{ij}^{**} - s_{ij}^*$ has been introduced. In addition, if there exists a shakedown state, it follows from the author's Equation [6] that

$$(\bar{r}_{ij} + s_{ij}^*)(\bar{r}_{ij} + s_{ij}^*) \leq 2k^2 \quad\ldots\ldots\ldots [3]$$

$$(\bar{r}_{ij} + s_{ij}^* + s_{ij}^\circ)(\bar{r}_{ij} + s_{ij}^* + s_{ij}^\circ) \leq 2k^2 \quad\ldots\ldots [4]$$

However, it will now be shown that the Inequalities [2], [3], and [4], herewith, are inconsistent.

The Inequality [4] can be continued in the form

$$2k^2 \geq (\bar{r}_{ij} + s_{ij}^*)(\bar{r}_{ij} + s_{ij}^*) + 2\, s_{ij}^\circ(\bar{r}_{ij} + s_{ij}^*) + s_{ij}^\circ s_{ij}^\circ = (\bar{r}_{ij} + s_{ij}^*)(\bar{r}_{ij} + s_{ij}^*) - 2|s_{ij}^\circ(\bar{r}_{ij} + s_{ij}^*)| + s_{ij}^\circ s_{ij}^\circ \quad\ldots\ldots [5]$$

The second step is valid, since in view of Inequality [2] of this discussion and the fact that the first term in the middle member of Equation [5] is nonnegative, the second term is less than zero. In view of Cauchy's inequality (following author's Equation [16]), Equation [5] leads to

$$2k^2 \geq (\bar{r}_{ij} + s_{ij}^*)(\bar{r}_{ij} + s_{ij}^*) - 2[(\bar{r}_{ij} + s_{ij}^*)(\bar{r}_{ij} + s_{ij}^*) \times s_{mn}^\circ s_{mn}^\circ]^{1/2} + s_{mn}^\circ s_{mn}^\circ = [\sqrt{s_{mn}^\circ s_{mn}^\circ} - \sqrt{(\bar{r}_{ij} + s_{ij}^*)(\bar{r}_{ij} + s_{ii}^*)}]^2 \quad\ldots\ldots [6]$$

Finally, it follows from Expressions [6], [2], and [3], herewith, that

$$2k^2 > [\sqrt{8k^2} - \sqrt{2k^2}]^2 = 2k^2 \quad\ldots\ldots [7]$$

which is an obvious contradiction. Thus the Inequality [1] is a necessary (although, in general, not a sufficient) condition for shakedown.

Author's Closure

Professor Hodge's contribution is appreciated. Unfortunately, as he states, the theorem he proves gives a necessary but not sufficient condition for shakedown. An interesting subject for future study is the relation between the theorems concerning the load limits for which shakedown just occurs, and those which recently have been proved concerning the loads at which unrestricted plastic flow occurs under steady proportional loading of an elastic perfectly plastic body.[3,4] In the case of continuous frame structures close analogies exist between the analytical methods for proportional loading and for variable, repeated loads.[5,6] It would be of great interest to find whether similar analogies can be utilized also in the case of continuous media.

[3] "The Safety Factor of an Elastic-Plastic Body in Plane Strain," by D. C. Drucker, H. J. Greenberg, and W. Prager, Paper No. 51—A-3, to be presented at the Annual Meeting, Atlantic City, N. J., Nov. 25–30, 1951, of The American Society of Mechanical Engineers.

[4] "Extended Limit Theorems for Structures and Continuous Media," by D. C. Drucker, W. Prager, and H. J. Greenberg, technical report, No. 59, Brown University to Office of Naval Research, March, 1951.

[5] "A Method for Calculating the Failure Load for a Framed Structure Subjected to Fluctuating Loads," by B. G. Neal and P. S. Symonds, *Journal of the Institution of Civil Engineers* (London), vol. 35, 1950–1951, pp. 186–198.

[6] "New Techniques for Computing Plastic Failure Loads of Continuous Frame Structures," by P. S. Symonds and B. G. Neal, paper presented at First U. S. National Congress of Applied Mechanics, Chicago, Ill., June 11–16, 1951, technical report No. 62, Brown University to Office of Naval Research, August, 1951.