Discussion

Evaluation of Stress Distribution in the Symmetrical Neck of Flat Tensile Bars[1]

E. H. Lee.[2] Presumably the type of strain distribution considered by the author applies only for a range of width-to-thickness ratios, for with flat wide specimens the constraint of the width prevents contraction, and necking occurs through the thickness of the specimen. Thickness gradients then become large, and the system no longer can be considered as in plane stress. It seems that there is a range of width-thickness ratios sufficiently large to permit the assumption of plane stress, and sufficiently small to avoid sharp necks producing local reduction in the thickness. Perhaps the author would comment on the range of validity of this type of behavior.

Author's Closure

The comments made by Professor Lee are appreciated. The author agrees, as was stated in the text, that the type of strain distribution considered can be applied only for a relatively narrow-range width-to-thickness ratio.

Presumably the type of sharp necks in the thickness direction that Professor Lee described for larger width-to-thickness ratios is the same type of necking described in the literature by Koerber, Siebel, Bijlaard, and Hill.

A separate investigation was carried out by the author to study what effect the width-to-thickness ratio and strain-hardening has on the mode of necking. These results were presented at the First U. S. National Congress of Applied Mechanics, June, 1951, at Chicago, Ill. The later results indicate that the mode of necking will undergo a transition from the symmetrical-type neck to the oblique neck as the width-to-thickness ratio is increased. However, there are not sufficient data available to make a general statement.

Free Vibrations of a Pin-Ended Column With Constant Distance Between Pin Ends[1]

N. J. Hoff.[2] The most important result obtained in the paper is an explanation of the failure of attempts to establish a nondestructive method of testing the elastic stability of structures. It often has been proposed that such structures should be loaded up to as high values of the loads as can be reached without danger of permanent deformation. At different load levels the lowest natural mode of vibration of the structure should be excited. The curve representing the natural frequencies plotted against the load can be extrapolated, and the buckling load of the structure is the load at which the natural frequency is zero.

The author has shown that in an ordinary column test the natural frequency depends greatly upon the amplitude of the vibration, and the frequency is considerably higher than zero at the instability load. As an example, it might be mentioned that when the amplitude is one half the radius of gyration of the cross section, the frequency at the buckling load is equal to 20 per cent of the natural frequency of a column upon which no end load is acting. This is true, naturally, only if the load is applied by means of controlled end displacements, as is done in an ordinary testing machine, and not through the application of dead weights. Consequently, extrapolation of the curve to zero frequency results in a load differing substantially from the critical load of the classical theory of stability. Moreover, variations in the amplitude of the vibrations at different stages of the loading cause a great deal of scatter in the results and add to the difficulties of obtaining the buckling load by means of vibration tests.

It appears, therefore, that attempts at determining the buckling load by such nondestructive tests either have to be given up or else will have to be modified in the light of the author's results.

E. H. Lee.[3] With reference to the author's comments on the restriction on the initial shape of the strut, it may be worth mentioning that this type of analysis will go through for an initial eccentricity comprising a sine curve with any integral number of half waves. While this provides an infinite number of initial configurations, they cannot be superimposed to cover any initial shape since the problem is nonlinear.

Vibration of Rectangular and Skew Cantilever Plates[1]

M. Z. v. Krzywoblocki.[2] As mentioned in the paper, the Rayleigh-Ritz method gives an upper bound, but it is possible to find a lower bound. The combination: The Rayleigh-Ritz and the Weinstein methods proved to be a successful mathematical tool, giving upper and lower bounds. It may be of interest to apply this combination in the case considered by the author.

Analysis of Deep Beams[1]

A. J. Durelli.[2] The problem dealt with here by the authors is very important in many industrial applications, and the contribution made by them will certainly be welcome since the study is far from exhausted.

It may be worth mentioning that a series of papers was pub-

[1] By Julius Aronofsky, published in the March, 1951, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 13, pp. 75–84.

[2] Professor of Applied Mathematics, Brown University, Providence, R. I.

[1] By David Burgreen, published in the June, 1951, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 73, pp. 135–139.

[2] Head, Department of Aeronautical Engineering and Applied Mechanics, Polytechnic Institute of Brooklyn, Brooklyn, N. Y. Mem. ASME.

[3] Professor of Applied Mathematics, Brown University, Providence, R. I. Mem. ASME.

[1] By M. V. Barton, published in the June, 1951, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 73, pp. 129–134.

[2] Professor of Gasdynamics and Theoretical Aerodynamics; Secretary, Panel on Fluid and Solid Mechanics, University of Illinois, Urbana, Ill.

[1] By H. D. Conway, L. Chow, and G. W. Morgan, published in the June, 1951, issue of Journal of Applied Mechanics, Trans. ASME, vol. 73, pp. 163–172.

[2] Supervisor, Armour Research Foundation of Illinois Institute of Technology, Chicago, Ill. Jun. ASME.

lished a short time ago along lines similar to those followed by the authors (1 to 6).[3] A comparison is made in these papers between the finite-difference method and the method of superimposing two stress functions in very much the same way as the authors did (2, 4). A photoelasticity test (5) was also conducted to check results. In other papers published shortly after (3, 6), the Galerkin method was used to approach the solution of the same problem, comparisons again were made, and tables were given for several loading conditions and widths of supports. Several of the numerical applications were made for beams loaded on the lower boundary, but the photoelastic test considers a beam loaded on the upper boundary. An interesting point brought out by this test is that the maximum stress is not on the axis of symmetry but on the lower boundary near the support.

A photoelastic test to check the authors' results does not seem too difficult to realize in the case of loading on the upper boundary. Uniform loading can be obtained either by using several sheets of cardboard (7), or by means of air pressure applied to a thin rubber hose placed inside a metal channel, with one side of the hose resting on the photoelastic specimen. The separation of the principal stresses should be easily obtained by iteration methods since all boundary conditions will be known and the shape is rectangular.

The related problem of the high beam on three supports, loaded on the lower boundary, has recently been approached using Fourier's series (8).

BIBLIOGRAPHY

1 "Vigas de Gran Altura Sometidas a Cargas Uniformes," by M. M. Guttero, *Ciencia y Tecnica*, vol. 107, July, 1946, pp. 1–55; August, 1946, pp. 81–134, and September, 1946, pp. 204–240.

2 "Sobre la Viga Simplemente Apoyada con Seccion Rectangular de Gran Altura," by E. Butty, *Ciencia y Tecnica*, vol. 109, October, 1947, p. 195.

3 "Solucion Variacional del Problema de la Viga Rectangular Simplemente Apoyada de Gran Altura," by A. Guzman and C. Luisoni, *Ciencia y Tecnica*, vol. 3, September, 1948, p. 119.

4 "Comentarios Sobre Vigas de Gran Altura Simplemente Apoyadas," by F. Garcia Olano and E. Fliess, *Ciencia y Tecnica*, vol. 3, November, 1948, p. 288.

5 "Ensayo Fotoelastico de una Viga de Gran Altura," by E. Sciammarella and M. Palacio, *Ciencia y Tecnica*, vol. 113, November, 1949, p. 249.

6 "Sobre la Viga Simplemente Apoyada de Gran Altura," by A. Guzman and C. Luisoni, *Ciencia y Tecnica*, vol. 114, 1950, p. 285.

7 "Distribution of Stresses in Partial Compression," by A. J. Durelli, Proceedings of the 13th Semi-Annual Eastern Photoelasticity Conference, June, 1941, p. 25.

8 "The Wall-Like Beam on Three Supports," by H. Parkus, *Oesterreichisches Ingenieur Archiv*, vol. 2, 1948, p. 185.

M. Z. v. Krzywoblocki.[4] The authors apply to the case of continuously distributed stresses the strain-energy method and a numerical method of finite difference and compare the results. But there does not exist any strain-energy method in the case considered by the authors. The sentence in the paper, "This solution by strain-energy methods would be exact only if an infinite number of parameters were used," cannot be accepted from a mathematical standpoint. Similarly, the conditions for a minimum V used by the authors are not, under any circumstances, sufficient conditions for a minimum of a function of n-variables. In the defense of the authors it should be mentioned that there exists a confusion in the literature on the subject, increased, perhaps, by the fact that one of the writers cited by the authors invented and published in one of his textbooks a proof of that method, which proof violates all the principles of elementary differential calculus and is no proof at all. It assumes as the base that the condition for a minimum of a function of n-variables is sufficient in the same form as accepted by the authors. But the conditions for a minimum of a function of n-variables are much more complicated. Moreover, we are seeking the absolute minimum of the function of n-variables and not a minimum. From the nature of the problem so defined, it is obvious which analysis must be used—calculus of variations. This question was pointed out in 1936 by Th. Pöschl in his brief and excellent but evidently unnoticed or forgotten paper in "Der Bauingenieur." Here are Pöschl's points:

1 The so-called strain-energy method is mathematically rigorous only in the case of concentrated loads.

2 From a mathematical standpoint it cannot be used under any circumstances in the case of continuously distributed stresses.

3 In the last case the methods of the calculus of variation should be used.

The writer published in 1947, two short papers on this subject in the *Journal of The Franklin Institute* and two letters in the *Journal of the Aeronautical Sciences* (under the titles, "On the So-called Principle of Least Work Method"). Recently, his paper with all details and with a proposition of the generalized Castigliano's principle appeared in *Oesterreichisches Ingenieur Archiv* (1951). In one of the examples it was shown clearly how large errors and completely wrong results may be obtained using the procedure called strain-energy method in the case of continuously distributed stresses.

The writer askes the authors of the paper to accept these remarks not as a criticism but as a certain contribution to the explanation of their results. The wrong conception of Castigliano's theorem should have been corrected earlier, since, in the case of continuously distributed stresses, the procedure used is a purely random-choice engineering process which in certain instances gave fair results. But it is impossible to foresee in which cases it will succeed and in which cases it will not. Hence it always should be verified by reliable methods (tests, etc.) and applied only to similar cases.

In case there are some fundamental changes in the system, the procedure again should be verified. Previous authors often supported their results based upon that procedure by experimental tests; but it does not mean that one may claim that the process always gives a solution (what is more, an exact solution). It is quite possible that tests will approve the results of the procedure in the present case justifying the authors' considerations.

The writer wishes that this would be the case. Even that fact should not be used as a justification for not presenting the truth to readers, which is that no item in this procedure has a mathematical justification, and one cannot talk about a minimum or exact solution, and that the cited author who developed the method simply misinterpreted the mathematical conception of the minimum of the strain-energy.

Closure by H. D. Conway

The comments of the discussers are appreciated.

The list of recent publications given by Dr. Durelli indicates the very considerable interest taken in the subject. Since the publication of the paper, further photoelastic tests have been made applying the load through various media (including cardboard) and quite good agreement with the theoretical results has been obtained.

In the first method of analysis used in the paper, the writing of the approximate stress function in the form of a series of polynomials leads, after equating the partial derivatives of the strain energy to zero, to a set of algebraic equations which yield

[3] Numbers in parentheses refer to the Bibliography at the end of this discussion.

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the parameters. An infinite number of parameters do not necessarily lead to an exact solution and this statement in the paper is retracted.

As the author understands it, Professor Krzywoblocki's criticism of the method is that the foregoing procedure does not necessarily lead to a minimum value of the expression for the strain energy. The writer believes that the following (suggested by Prof. J. N. Goodier) shows that it does.

The strain energy V may be written as the integral of

$$V_0 = \frac{1}{2E}\left\{\sigma_x{}^2 + \sigma_y{}^2 - 2\mu\sigma_x\sigma_y + 2(1+\mu)\tau_{xy}{}^2\right\}$$

where the stresses are such that V is stationary ($\partial V/\partial n = 0$) for the set of variations we admit. Assuming that the stresses are varied to $\sigma_x + \delta\sigma_x$, $\sigma_y + \delta\sigma_y$ and $\tau_{xy} + \delta\tau_{xy}$, respectively, and that equilibrium and the original boundary conditions are maintained, the corresponding strain energy is the integral of

$$\begin{aligned}\frac{1}{2E}\{&\sigma_x{}^2 + \sigma_y{}^2 - 2\mu\sigma_x\sigma_y + 2(1+\mu)\tau_{xy}{}^2 \\ &+ 2\sigma_x\delta\sigma_x + 2\sigma_y\delta\sigma_y - 2\mu(\sigma_x\delta\sigma_y + \sigma_y\delta\sigma_x) + 4(1+\mu)\tau_{xy}\delta\tau_{xy} \\ &+ (\delta\sigma_x)^2 + (\delta\sigma_y)^2 - 2\mu\delta\sigma_x\delta\sigma_y + 2(1+\mu)(\delta\tau_{xy})^2\} \\ &= V_0 + \delta V_0 + \text{value of } V_0 \text{ for } \delta\sigma_x, \delta\sigma_y, \delta\tau_{xy}\end{aligned}$$

Since δV_0 (integrated) is zero and the third term is always positive, we have the conditions for a minimum.

Further comment seems unnecessary. Suffice it to say, that despite the dire consequences predicted by Professor Krzywoblocki, the writer still believes the foregoing to be a very valuable method.

Large-Deflection Theory for Plates With Small Initial Curvature Loaded in Edge Compression[1]

Samuel Levy.[2] The analysis of plates with mixed boundary conditions, that is, displacement conditions on one pair of edges and stress conditions on the other, is often necessary in structural design but is generally difficult to make. The author has managed to obtain such solutions with the additional complications of initial curvature and large deflections. Although his primary object was to evaluate various methods of deducing buckling loads from measured center deflections and strains, the methods he has developed for taking account of mixed boundary conditions are quite general and will be useful in analyzing other plate problems.

The author develops an "exact" solution for several examples of buckling of a square plate, and from the resulting plots of deflection and strain versus load, draws interesting conclusions regarding five widely used approximate methods of estimating buckling loads from test data. The comparison of experimental results with the computed curves confirms the theoretical derivations and shows the important effects both of initial deviations from flatness and of restraint of the supported edges from transverse displacements.

The writer believes that it will be necessary to use large-deflection theories to explain satisfactorily a large proportion of the secondary effects associated with the bending of plates. The importance of even small deviations from flatness, as evidenced by the author's Fig. 4, was first brought to light by Hu, Lundquist, and Batdorf. The additional importance of the absence of restraint (in plane warping) along the supported edges has never been so clearly shown as in the comparison of Figs. 2 and 3 in the paper. The stress distribution in the plate at loads of about 1.7 times the buckling value is markedly different in the two cases.

S. B. Batdorf[3] and G. J. Heimerl.[3] Experimental buckling stresses are obtained mainly for three purposes, as follows:

1 To check theory.
2 To indicate the behavior of actual structures.
3 To provide a basis for calculating postbuckling behavior.

The paper is concerned with an evaluation of current procedures for predicting experimental buckling loads for the purpose of checking theories, which are usually concerned with the behavior of perfectly constructed and perfectly loaded specimens. Accordingly, the paper deals with the determination of the buckling loads of flat plates from measured behavior of imperfect plates. The analysis is based upon boundary conditions of uniform displacement of the loaded edges and stress-free supported edges. In these respects, it differs from NACA Technical Note 1124 which was concerned with a comparison of the behavior of imperfect plates with that of perfect plates when all edges were kept straight. It would appear that the author's boundary conditions are appropriate for a single plate and for a variety of more complex test specimens, such as rectangular tubes, for example, whereas the boundary conditions of the Technical Note mentioned are more appropriate for structures such as stiffened panels in which the continuity of the sheet helps to hold the edges of each constituent plate straight.

With regard to methods for predicting flat-plate buckling loads experimentally, the author showed that the point of inflection of the curve of load versus deflection gives the true buckling load more accurately than other methods which have been used. Reference to the curves of TN 1124 indicates that this is true also when the side edges of the plate are kept straight. It is generally recognized, however, that slopes cannot be reliably determined when test data exhibit appreciable scatter. It would seem that this remark should apply a fortiori to the determination of the point of inflection, which in effect represents a determination of the second derivative of the curve, whereas the slope constitutes only the first derivative. Moreover, if one is concerned with either plastic buckling or with buckling with a stress close to the elastic limit of the material, there actually may be no point of inflection. For these two reasons it might be expected that the vertical tangent of the plot of load against axial median strain at the center of the plate would generally be preferable to the inflection-point method. However, it should be noted that this method is of limited applicability; whereas the load versus median axial-strain curve exhibits a vertical tangent when the sides of the plate are stress-free, no vertical tangent occurs when the sides are kept straight.

As the author has indicated, the top-of-the-knee method will give somewhat lower buckling loads. It may be in order to point out here that the top-of-the-knee method has been used extensively by the NACA for checking flat-plate theory, but it has been applied only to cases in which there was negligible initial curvature, for example, extruded H-, Z-, and channel sections, and drawn square tubes. Consequently, the knee of the curve is so sharp that little latitude actually exists for choice of buckling stress, and the experimental buckling loads can be considered to be a close indication of the flat-plate buckling load.

According to both the author's results and those of TN 1124, initial eccentricity influences plate behavior markedly in the vi-

[1] By J. M. Coan, published in the June, 1951, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 73, pp. 143–151.

[2] Aircraft Structures Group, U. S. Department of Commerce, National Bureau of Standards, Washington, D. C. Mem. ASME.

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