$F$, $m$, and $a$ may be chosen so that $g_0$ is unity and may be omitted from the equations. Thus either form ($a$) or ($d$) may logically be used, depending upon whether one prefers to work in a system of units where $g_0$ and $J$ are unity, or are not unity, respectively. The authors feel that form ($a$) is preferable because of its simplicity.

The use of the stagnation enthalpy, stagnation temperature, and stagnation pressure for the case of variable specific heat as well as constant specific heat, as suggested by Professor Hall, is of course acceptable. However, the authors felt that simplicity in making numerical calculations should be a major aim of the analytical formulation. After having considered originally the use of the stagnation parameters, the conclusion was reached that they entailed unnecessary complications in the calculations, and the method described in the paper was adopted.

In reply to Professor Hall's comments on the temperature at which the heat of reaction is reckoned, it might be stated that here again the aim was simplicity in numerical calculations, and that in this respect the methods given in the paper were felt preferable to the selection of the heat of reaction at a fixed base temperature.

We think Professor Rudnick's comments on the choking phenomenon interesting and illuminating.

As to the possibility of beginning the integration of Table 1 at the $\mathsf{M} = 1$ surface, it is agreed that this is convenient in many cases. There are, however, instances in which this surface can be located only by trial and error, in which case the method of the paper seems preferable. The choice of the method seems to depend on the physical nature of the problem, and it is doubtful whether any general rule can be laid down.

# Sliding Friction of Ball Bearings of the Pivot Type[1]

W. D. Anderson.[2] During the last few years considerable progress has been made by the ball-bearing industry in developing ultrasensitive bearings for instrument work, and, while most of this progress has been based upon an experimental approach, the value of a mathematical analysis should be self-evident.

The assumptions on which the analysis is based are reasonable and necessary to avoid prohibitive mathematical complications. However, certain of these assumptions should be reexamined before attempting to reduce to practice the results of the analysis.

The assumption that the "pivot and race surfaces are correctly centered" implies that they also have colinear axes. In practice, errors of manufacture and assembly usually produce some angular error between the axes of the pivot and the race. In the general case of a race whose surface is a torus rather than a sphere, such alignment errors will cause differential ball speeds, thus violating the assumed symmetrical ball distribution and causing the balls to contact each other or to contact the ball separator. In either case, frictional torques will be imposed upon the pivot.

Experimental results indicate that these frictional torques are much greater than the torque due to imperfect rolling between ball and race. These frictional torques may be reduced materially by special designs of separators to space the balls. However, they still constitute a major problem in the design of ultrasensitive ball bearings.

[1] By H. Poritsky, C. W. Hewlett, Jr., and R. E. Coleman, Jr., published in the December, 1947, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 69, p. A-261.

[2] Assistant to Chief Engineer, Norma-Hoffmann Bearings Corporation, Stamford, Conn.

Thomas Barish.[3] In 1938 an attempt was made to calculate and measure the ball-race friction in radial ball bearings under radial loads just as the authors of this paper have done for pivot bearings under angular contacts. This work was done by Seymour Herwald[4] as a thesis for a degree at the Case Institute of Technology. Most of the conclusions closely agree with those of this later work but others were quite at variance; particularly, an entirely different concept of the "slipping" was deduced and confirmed by other considerations.

The calculations followed similar lines except that under pure radial load (1) there is no tendency for only one race to slip; the friction of both races was calculated although not as completely as in this later work; (2) the friction was much smaller, being due only to the difference in radii along the contact area. True rolling occurs at points $A,A$, Fig. 1 of this discussion. In between, the ball surface precedes, and outside of these points the race surface precedes. These two effects balance on the neutral points $A,A$, and the friction torques about these points were summarized.

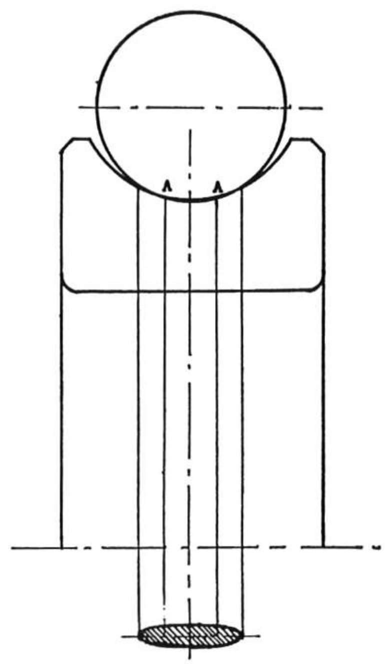


Fig. 1

Tests were made so as to eliminate all cage friction and most variations due to inequality of loading, as well as ball and race inaccuracies. A heavy pendulum was suspended from a single bearing with only two balls. The loss of amplitude for many swings was measured by an indicator at the end of the stroke, Fig. 2. It is evident that this friction will require fairly heavy loads to give appreciable widths to the contact area. Tests were made with varying loads and race curvatures.

The calculations and tests indicated a coefficient of "sliding friction" of 0.2, approximately. The present paper assumes the same coefficient and reports agreement of calculations and tests. However, the figures of Mr. Herwald added friction at both inner and outer rings, whereas the new figure is only the friction at the inner ring.

Mr. Herwald and the writer believed that there was no actual

[3] Consulting Engineer, Washington, D. C.

[4] Westinghouse Electric Corporation, East Pittsburgh, Pa. This work was done with the co-operation and the equipment of the Marlin Rockwell Corporation, Jamestown, N. Y.

sliding at the contact surfaces (except possibly at the very edges where unit pressures are low and some sort of lubrication can exist). In most of the contact area, pressures vary from 50,000

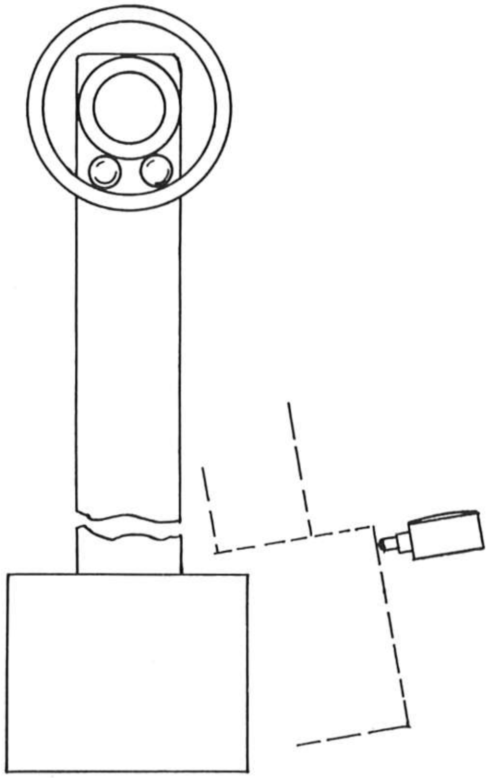

FIG. 2

to 400,000 psi, and it was thought inconceivable that the metal would actually slide under such pressures and for such short displacements. Instead it is believed that the surface and subsurface metal are displaced tangentially; that in the section where

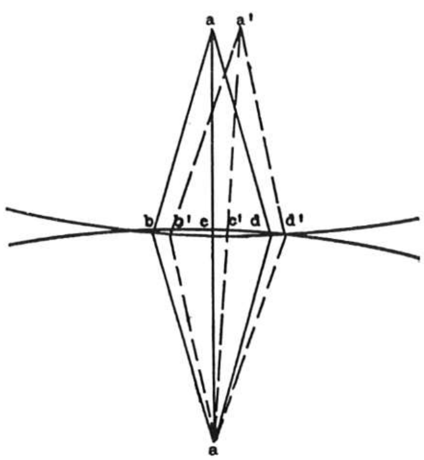


FIG. 3

sliding is supposed to occur, the lines *ab*, *ac*, and *ad*, Fig. 3, shift to the dotted lines *ab'*, *ac'*, and *ad'*.

True sliding might occur if the displacements were large, but it would then resemble tearing rather than sliding. Under such conditions ball bearings are rapidly destroyed, as under very bad off-square operation when there is definite sliding between ball and race. Similarly, too close curvatures under heavy loads or steep angles will show rapid failure.

If actual sliding did occur, the friction would be erratic with the continual change from static to kinetic friction.

The absence of actual sliding is further confirmed by a similar action in metal-to-metal friction drives. Here the same type of contact delivers a positive tangential force without slipping, as long as the normal force is large enough. Instead of slipping, which would destroy the surfaces quickly, there is a definite and constant creep, corresponding to the displacement of the lines in Fig. 3, and varying with the size of the tangential force.

Fig. 1 of the original paper indicates how banding of the balls would occur if there were sliding, and only at the outer race. In practice, banding does occur. It always shows up under the following conditions: Enough thrust load so that the top balls are not loose, and no disturbance or removal of the load. These conditions happen regularly on ball-bearing testing machines.

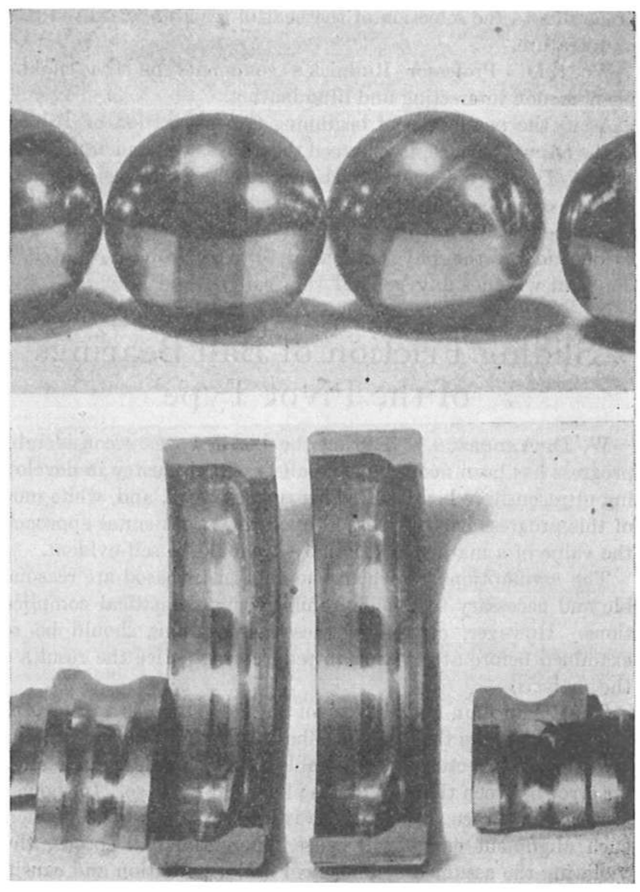

FIG. 4

The banding is clearly shown in Fig. 4 herewith. The banding is exactly on a diameter. The width is that of the wider area, the inner-ring contact, and does not follow the theory propounded in Fig. 1 of the paper.

This paper considers only pivot bearings, and these may show different results. A simple test is suggested; the conditions proposed would give a higher ball speed or cage speed around the shaft. This can be measured accurately since it is accumulative, i.e., count the number of cage revolutions per 100 shaft revolutions.

The paper mentions the fact that the deflection in the bearing under radial load changes as the bearing rotates. The deflection is less when there is a ball at the bottom compared with the race straddling two balls. Any slight looseness in the bearing produces the same effect and adds to the inequalities of the deflection. Both of these effects fall off rapidly with increase in the number of balls as shown in Fig. 5 of this discussion. So much is this so that it would be an error to make a sensitive pivot bearing with less than 5 balls, and even a 5-ball bearing may be questionable.

In most pivot ball bearings and especially in sensitive gyros,

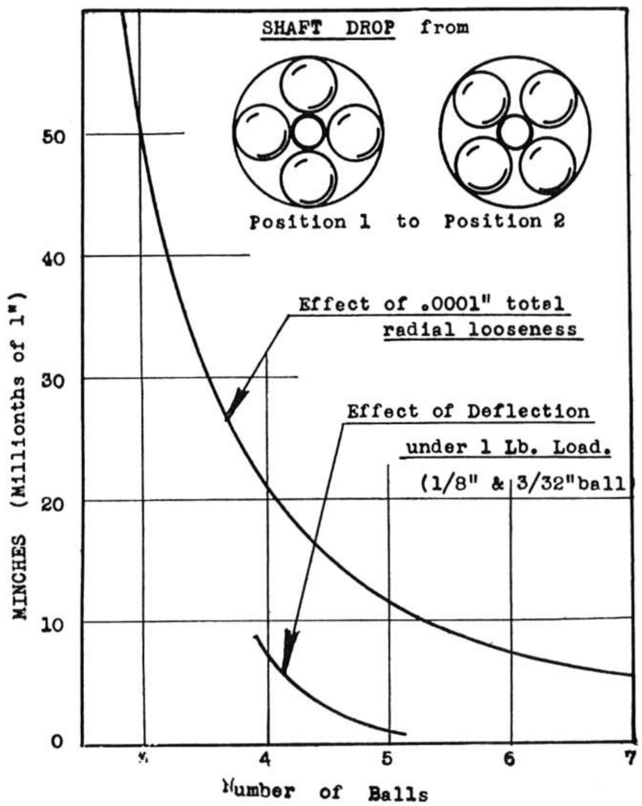


Fig. 5

to have the friction constant is as important as having it low. Hence we should note in this type of development that the ball-to-ball or ball-to-cage friction frequently exceeds the ball-to-race friction, and is very erratic.

A. B. Jones.[5] The authors of this paper have presented a clear and well thought out analysis of the effect of contact area spinning torque on the friction of pivot-type ball bearings. Their derivation of the torque required to spin an elliptical pressure area about an axis normal to the center of the pressure surface agrees identically with the results arrived at by the writer.

In addition to the spinning action, there is evidence that the slippages due to the curvature of the pressure surface itself are also of prime importance; in fact, in the conventional radial-type ball bearing under radial load, slippage of this type is the only source of contact-area friction. That slippage of this type may also be appreciable in pivot-type ball bearings is indicated by the fact that the authors obtained correlation with minimum values of torque tests using a coefficient of friction of 0.20.

Experiments conducted by the writer, in which a ball was pressed on opposite sides by sections of curved raceways and the ball rotated about an axis through both contact points, indicate a maximum value of 0.14 for the coefficient of friction.

J. R. Macintyre[6] and E. E. Lynch.[6] It is always interesting to compare theoretical calculations with measured values. In this particular case it is probably impossible to separate sliding friction from the other types of friction which enter into the turning of ball bearings, by means other than theoretical. The binding of balls as they become partially "jammed," due to unavoidable tolerances in mechanical construction, and the unbalance torques exerted by the rotating masses due to the non-perfect geometry of the bearing parts, makes the friction a variable which is difficult to measure. Moreover, the breakaway friction torques differ from the running torques; and running friction torques vary with the speed. It is interesting to note, therefore, the close agreement which is indicated between theoretical sliding friction and the minimum values of precise measurements of total friction.

The theoretical approach can be used to save time when comparing friction (sliding) of different shaped bearings. It may even indicate the advantages of new materials for bearings. It certainly sets a minimum friction for a particular design which can be used as a guide in determining the "point of diminishing return" on efforts made to reduce manufacturing tolerances.

In publishing this work the authors have shown the way toward calculation of total friction. This is a goal which, if attained, will save much time-consuming effort in tests and statistical interpretation of the results which is often necessary because of the extremely small torques involved.

Bryce Ruley.[7] This excellent paper treats a rather special case of sliding friction in ball bearings and, for that reason, the sliding friction due to the torque $\omega_1$ can be isolated from other frictional effects.

Earlier investigators, notably A. Palmgren, in a series of papers,[8] in 1926 and 1928, considered the effect of sliding friction in the contact areas between balls and raceways. The case considered was the more general one where curvature within the contact areas could not be neglected, and the conclusions reached, while in general agreement with the present paper, were not as specific and easily verified by tests as in the special case of pivot bearings. Some cases were calculated in the third of the Palmgren papers, and reasonable confirmation by experiment was found even when friction due to curvature of the contact areas was not small.

The occasional occurrence of contact bands on balls in bearings of standard types (i.e., deep-groove ball bearings and angular-contact ball bearings), usually when the load is predominantly thrust or when all internal looseness has been removed from the bearing during mounting so that the balls never pass through an unloaded zone in the bearing, indicates that there may be cases where the $\omega_1$ torque is of predominant importance even in bearings of these standard types.

[5] Chief Research Engineer, New Departure Division, General Motors Corporation, Bristol, Conn.

[6] Works Laboratory, General Electric Company, West Lynn, Mass.

[7] Senior Engineer, SKF Industries, Inc., Philadelphia, Pa., Jun. ASME.

[8] "The Nature of Rolling Resistance," "Investigations With Regard to Rolling Under Tangential Pressure," and "Sliding Friction in Ball Bearings," *The Ball Bearing Journal*, published by SKF Industries, Inc., Philadelphia, Pa.

Z. Horak.[9] Starting from the Hertz's theory, the writer has deduced,[10] for the moment of slipping friction, the expression

$$D = \mu \frac{3P}{2\pi ab} \int \int \sqrt{x^2 + y^2} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dxdy$$

which is a direct consequence of the authors' Equations [12] and [15]. The integration is over the ellipse with semiaxes $a$, $b$ and has been carried out by the meter,[10] p. 163, as follows

We put

$$\frac{x}{a} = \rho \cos \varphi, \quad \frac{y}{b} = \rho \sin \varphi$$

and obtain

$$D = \mu \frac{3P}{2\pi} \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=1} \sqrt{\rho^2 a^2 \cos^2\varphi + \rho^2 b^2 \sin^2 \varphi}$$

$$\sqrt{1 - \rho^2}\, \rho d\rho d\varphi = \mu \frac{3P}{2\pi} \int_0^1 \rho^2 \sqrt{1 - \rho^2}\, d\rho \int_0^{2\pi} \sqrt{a^2 - (a^2 - b^2) \sin^2 \varphi}\, d\varphi$$

Now writing

$$\rho = \sin \tau$$

we have

$$\int_0^1 \rho^2 \sqrt{1 - \rho^2}\, d\rho = \int_0^{\pi/2} \sin^2 \tau \cos^2 \tau\, d\tau = \frac{1}{8} \int_0^{\pi/2} (1 - \cos 4\tau) d\tau = \frac{\pi}{16}$$

and

$$\int_0^{2\pi} \sqrt{a^2 - (a^2 - b^2) \sin^2 \varphi}\, d\varphi = 4a \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi = 4a\, E(k)$$

where

$$k^2 = \frac{a^2 - b^2}{a^2}$$

Therefore we get [10] p. 164 (see author's paper, Equation [17])

$$D = \frac{3}{8} E(k) a \mu P.$$

## Authors' Closure

The authors wish to thank the discussers for their interesting comments and the valuable contribution of their thoughts and experience on the subject of ball-bearing friction, and they believe that the discussions enhance greatly the value of the paper and bring out more clearly some valuable points which, due to limitation of space and other causes, were passed over rather hurriedly.

As pointed out by Messrs. Anderson, Barish, MacIntyre, and Lynch, and Ruley, the assumptions made by the authors were of the most ideal type. They correspond to the *simplest possible* conditions, and the analysis based upon them leads to the *smallest possible* friction which the bearings can have. Since in practice these conditions are seldom satisfied, it is clear that the actual friction obtained will be both larger and variable. In practice neither the pivot nor the race will be a true surface of revolution, and they will be slightly misaligned; the balls' radii will differ slightly, nor will the balls be truly spherical. There will thus be friction due to the various geometric imperfections and misalignments; there will be further friction due to the ball separator if there is one, or between the individual balls in case there is no ball separator. If the balls are of slightly different size, they will travel at different speeds about the axis and will push up against each other, thus creating even more friction. Furthermore, even with perfect geometry, the ball centers may deviate from the ideal circular path, and the balls may possibly wedge themselves into a narrow space. This is conceivable since the centers of the two contact areas of a ball need not be at diametrically opposite points on the ball—the line joining them may make any angle with the radial direction at either center which is less than the angle of friction. There are further friction effects due to squeezing out the oil film, and due to "solid damping" (the latter makes the balls always run "up-hill,") as well as effects due to particles of ordinary plain dirt. These are some of the neglected items which are responsible for increasing the actual pivot frictions.

Returning even to the ideal assumptions of the paper, as pointed out in connection with Figs. 1 to 3 of the paper, it is assumed there that the contact areas have negligible curvature. Thus on Fig. 3 of the paper it has been assumed that line $A$, $A_1$ is a straight line. Actually this is not the case and a very slight amount of slipping over the pivot contact will occur due to its curvature. This effect was termed a "second-order effect," and, as stated in the paper, was neglected. As pointed out by Messrs. Barish, Jones, and Ruley, for the case of radial bearings, these effects predominate. When the theory of the paper is applied to such radial bearings, so long as the curvature of the contact area is neglected, the contact area is flat, and *no slipping* need occur over the contact with either the inner or outer race, and the computed friction torque vanishes. However, if the curvatures of the contact area are appreciable, no true rolling of the ball, say, over the stationary race, can take place; instead, there is a tendency for slipping in opposite directions in the middle and outer portions of the contact band, as shown on Fig. 1 of the Barish discussion. It is gratifying to know that a study of this effect has been made by Messrs. Herwald and Barish; it is highly desirable to have the results of such studies more widely publicized by presenting them at meetings of technical societies, and having them published, so that they will be available to the engineering profession. As noted in the paper, this effect may occur even for bearings of the pivot type, though it will not be very pronounced unless the curvatures of the pivot and the balls are very close to each other, whereupon the area of contact and the resulting stresses are not given too accurately by the Hertz theory, and the whole analysis requires substantial corrections.

An interesting point brought out by Mr. Barish relates to the question of slipping over the contact areas. It may very well be, as pointed out by Mr. Barish, that over the inner part of the contact ellipse between the ball and the race, where the prescribing slip is small and where the possible friction force is high (due to the high normal pressure) the inner portions of the contact areas lock and twist. Over the peripheral portions of the contact areas slipping certainly does take place. This is an interesting phenomenon which deserves further study and experimentation. The analogous phenomenon of creep of a wheel over a rail has been studied by Carter,[11] but the authors know of no corresponding study for the ball contact problem.

However, the authors believe that even when this aspect of the

[9] Professor of Technical Physics, Technical University, Prague, Czechoslovakia.

[10] "Theory of Slipping Friction," by Z. Horak (in Czech), Proceedings of the Masaryk Academy of Work, vol. 18, Prague, 1944, pp. 150, 169.

[11] "On the Action of a Locomotive Driving Wheel," by F. W. Carter, Proceedings of the Royal Society, London, England, series A, vol. 112, 1926, p. 151.

contact phenomena has been taken into account it will not lead to any substantial changes in the conclusions of the paper regarding the geometric analysis of the ball motion or the friction torque, computed on the basis of slip motion. Referring to Figs. 1 to 3 of the paper, it is a geometric fact that no matter how the ball pivots, the four circles on the ball through $A$, $A_1$, $C$, $C_1$ on the ball at the edge of the contact bands and the corresponding circles over the pivot through $A$, $A_1$ and over the race through $C$, $C_1$ cannot all have the same length ratios. Since at these extreme points of the contact bands (through $C$, $C_1$) slipping must take place, the main conclusions obtained must be valid.

The predicted ball motion has been verified experimentally, as stated in the paper, by rotating the outer race in the direction opposite to the pivot and at such a speed that the ball center was maintained at rest, and observing the motion of the ball with a low-power microscope. The rotation diameter of the ball was found by attaching small magnets to the ball surface.

Again the torque was calculated by multiplying the local Hertz pressure by $\mu$ and by the lever arm about the pivot axis. This moment for the ball-pivot contact is larger than for ball-race contact; since the actual friction force may be less than the maximum value used, the conclusion was drawn that the actual friction moment is represented by the smaller of two moments, and that slipping will occur between the ball and the race. A more complicated motion at ball-race contact may occur—partly slipping, part twisting and untwisting—but this does not change the magnitude of the computed maximum friction torque. The authors doubt whether the friction torque which would be computed from the complete theory when the latter becomes available would be appreciably less than the torque computed in the paper. They cite the observed agreement with the *minimum* measured friction torque as a partial proof of this belief.

It is pointed out in the Barish discussion, for the case of radial bearings one must add the friction between the ball and inner race as well as between the ball and the outer race. Contrariwise, in the paper such an addition was not carried out. The reason is very clear. If no slipping takes place between the ball and the pivot no energy can be lost in friction at that contact. Moreover, the friction torque over the contact between the ball and the race must be transmitted in turn to the contact between the ball and the pivot, by Newton's third law of equal action and reaction (neglecting gyroscopic effects for the balls for rapid rotation.) In the radial bearing case the slipping is of an entirely different type and rotation takes place about the points $A$, $A$ of Fig. 1 of the Barish discussion, and about two similar points over the contact with the outer race.

In conclusion the authors thank Mr. Horak for the much shorter derivation of the final equation for the friction torque.

# On the Use of Power Laws in Stress Analysis Beyond the Elastic Range[1]

S. B. BATDORF[2] AND E. Z. STOWELL.[2] This paper shows that although desirable from the point of view of simplicity, a power law for the stress-strain relation in the theory of plastic deformation must be used with caution. The authors also show that the simplifications occur only if a single power law is applied over the whole range. However, the particular example chosen to illustrate these points may result in an unduly pessimistic impression as to the possibility of handling problems with satisfactory approximation by use of a power law. The purposes of this discussion are to show that much better results can be achieved using a different law, and to call attention to certain considerations which may serve as a guide in the proper selection of the power law to be used.

One should have a rough idea of the range of the stress-strain relation to be approximated. In the example of the paper, the power law used is shown as a dotted line, and the more realistic stress-strain curve as the solid line in Fig. 1 (the dashed curve will be discussed later). We have added, as points $C$ and $D$ (Fig. 1 of this discussion) the highest stresses occurring when the plastic domain extends to 10 $r_0/9$ and 4 $r_0/3$, respectively. It is clear that the dotted curve is a very bad approximation to the applicable portion of the solid curve in the first case and not too good in the second.

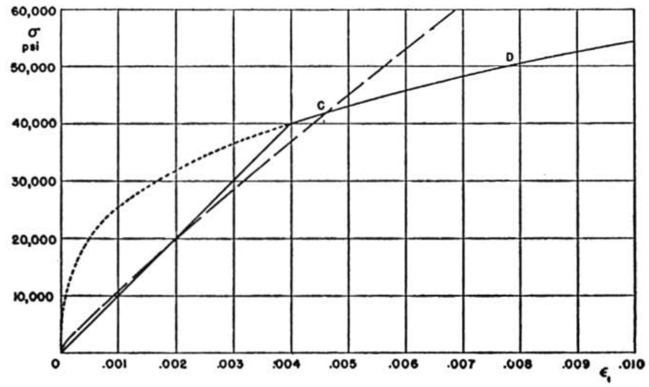


FIG. 1

In the second place it is desirable to assess the relative importance of good approximation in the elastic and plastic portions of the stress-strain curve. This can be done very roughly beforehand, at least in such cases as the example of the paper. To do this, let us assume initially that the problem is a purely elastic one. One may now ask, what fraction of the applied load is absorbed in the immediate neighborhood of the applied force, i.e., $r_0 < r < 10/9\, r_0$ or $r_0 < r < 4/3\, r_0$?

To answer this question we take $m = 0$ in Equations [20] and [25] of the paper and obtain $\alpha = 2$, so that from Equation [21]

$$S_1 = \frac{\sigma_1}{p} = -(r_0/r)^2$$

or

$$\sigma_1 = -p\left(\frac{r_0}{r}\right)^2$$

The radial load at the distance $r$ is $2\pi r\,\sigma_1$; it follows that the initial load has dropped only 10 per cent at $r = 10\, r_0/9$ and 25 per cent at 4 $r_0/3$. In the actual case, the regions under consideration are in a plastic state and therefore support less than the 10 per cent and 25 per cent of the load just computed elastically. Since most of the load is absorbed elastically, it would seem more important to obtain a fair fit to the elastic than to the plastic part of the stress-strain curve.

Applying these thoughts, a calculation was made on the assumption that

$$\sigma_i = 45{,}000\left(\frac{e_i}{0.005}\right)^{0.9}$$

which is given as the dashed curve in Fig. 1 of the present discussion, and is a fair fit to the solid curve up to point $C$. The results of the application of this law are shown in Figs. 2 and 3 of this discussion, which correspond to Figs. 2 and 3 of the paper. The

[1] By Alice Winzer and W. Prager, published in the December, 1947, issue of the JOURNAL OF APPLIED MECHANICS, Trans. ASME, vol. 69, p. A-281.

[2] Langley Memorial Aeronautical Laboratory, Langley Field, Va.